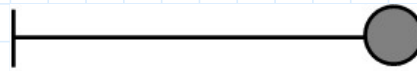


Dinamika konstrukcija

Primeri

Novi Sad, 2026. godina

Primer 1. Slobodne neprigušene vibracije

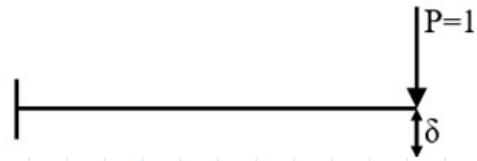


Podaci (kretanje mase samo u vertikalnom pravcu; 1 stepen slobode kretanja):

$$E_m := 30 \cdot 10^6 \quad L := 3 \quad b := 0.3 \quad d := 0.6 \quad m := 20 \quad v_0 := 0.5 \quad y_0 := 0.01$$

Fleksibilnost i krutost dinamičkog modela:

$$I := \frac{b \cdot d^3}{12} = 5.400 \cdot 10^{-3}$$



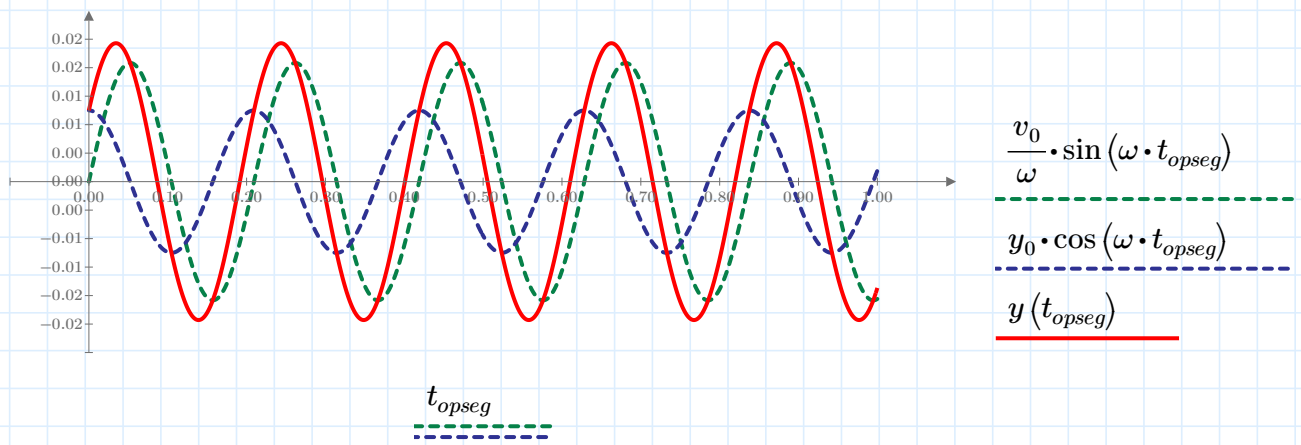
$$\delta := \frac{L^3}{3 \cdot E_m \cdot I} = 5.556 \cdot 10^{-5} \quad k := \frac{1}{\delta} = 1.800 \cdot 10^4$$

Svojstvena kružna frekvencija: $\omega := \sqrt{\frac{k}{m}} = 30.00$ Period svojstvenih vibracija: $T := \frac{2 \cdot \pi}{\omega} = 0.209$

Svojstvena frekvencija: $f := \frac{1}{T} = 4.775$ Tehnička frekvencija: $n := 60 \cdot f = 286.479$

Jednačina kretanja: $y(t) := \frac{v_0}{\omega} \cdot \sin(\omega \cdot t) + y_0 \cdot \cos(\omega \cdot t)$

Grafik: $t_{pocetak} := 0 \quad t_{korak} := 0.001 \quad t_{kraj} := 1.0 \quad t_{opseg} := t_{pocetak}, t_{korak} \cdot t_{kraj}$



$$opseg := \text{vec}(t_{opseg}) \quad nizVrednosti := \overrightarrow{y(\text{vec}(t_{opseg}))} \quad \begin{aligned} \max(nizVrednosti) &= 0.01944 \\ \min(nizVrednosti) &= -0.01944 \end{aligned}$$

solve, t
assume, $0 < t < 0.1$
float, 6

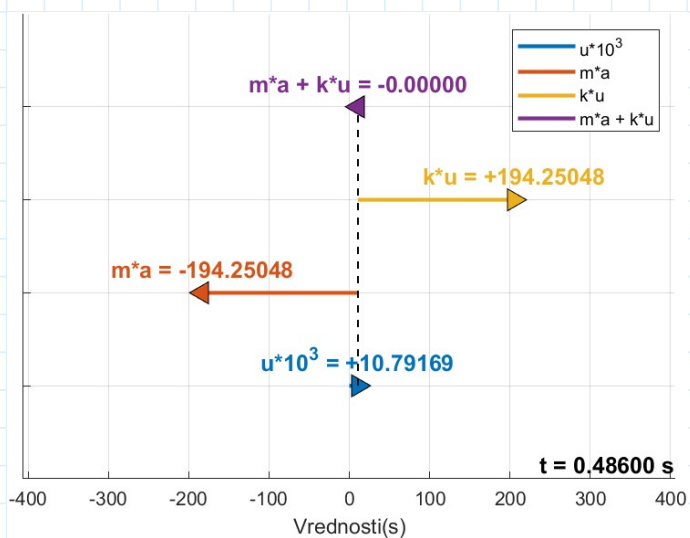
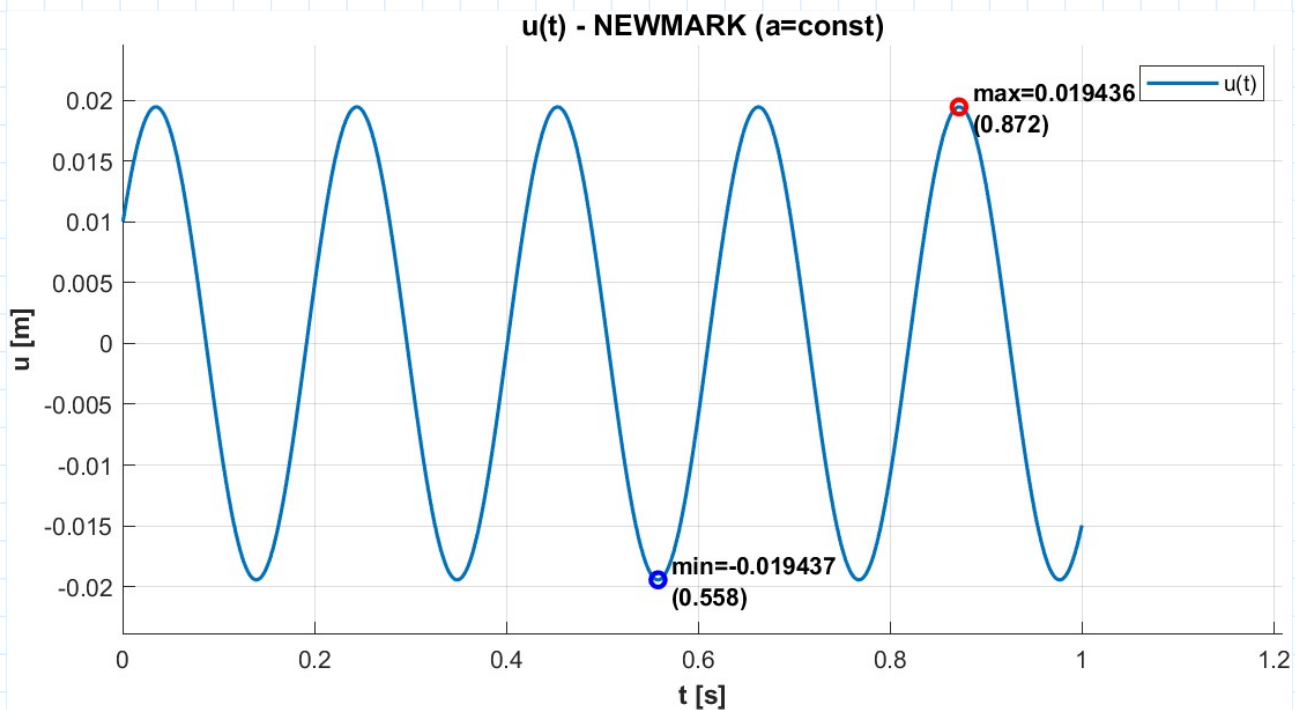
$$t_{ext} := \frac{d}{dt} y(t) = 0 \longrightarrow 0.0343459 \quad y_{ext} := y(t_{ext}) = 0.01944$$

Amplituda pomeranja: $C := \sqrt{\left(\frac{v_0}{\omega}\right)^2 + y_0^2} = 0.01944$ Fazna razlika: $\alpha := \text{atan}\left(\frac{y_0 \cdot \omega}{v_0}\right) = 0.540$

Trenutak dostizanja prve amplitude: $t_{max_1} := -\frac{\alpha}{\omega} + \frac{\pi}{2 \cdot \omega} = 0.03435$

Amplituda pomeranja: $y_{max} := C \cdot \sin(\omega \cdot t_{max_1} + \alpha) = 0.01944$

Numerička integracija (Matlab)



```

m = 20.00000 [t]
k = 18000.00000 [kN/m]
c = 0.00000 [t/s]
ceta = 0.00000
u0 = 0.01000 [m]
v0 = 0.50000 [m/s]

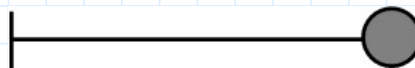
w = 30.00000 [rad/s]
T = 0.20944 [s]

umax=0.0194362555[m] (u t=0.87200[s])
umin=-0.0194365047[m] (u t=0.55800[s])

vmax=0.5830923026[m/s] (u t=0.40100[s])
vmin=-0.5830935963[m/s] (u t=0.71500[s])

amax=17.4928542529[m/s/s] (u t=0.55800[s])
amin=-17.4926299898[m/s/s] (u t=0.87200[s])
    
```

Primer 2. Slobodne prigušene vibracije



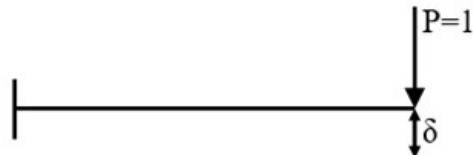
Podaci (kretanje mase samo u vertikalnom pravcu; 1 stepen slobode kretanja):

$$E_m := 30 \cdot 10^6 \quad L := 3 \quad b := 0.3 \quad d := 0.6 \quad m := 20 \quad v_0 := 0.5 \quad y_0 := 0.01$$

$$1) c_1 := 60 \quad 2) c_2 := 1440$$

Fleksibilnost i krutost dinamičkog modela:

$$I := \frac{b \cdot d^3}{12} = 5.400 \cdot 10^{-3}$$



$$\delta := \frac{L^3}{3 \cdot E_m \cdot I} = 5.556 \cdot 10^{-5} \quad k := \frac{1}{\delta} = 1.800 \cdot 10^4$$

$$\text{Svojstvena kružna frekvencija: } \omega := \sqrt{\frac{k}{m}} = 30.00 \quad \text{Period svojstvenih vibracija: } T := \frac{2 \cdot \pi}{\omega} = 0.209$$

1) Analiza za $c_1 = 60.00$

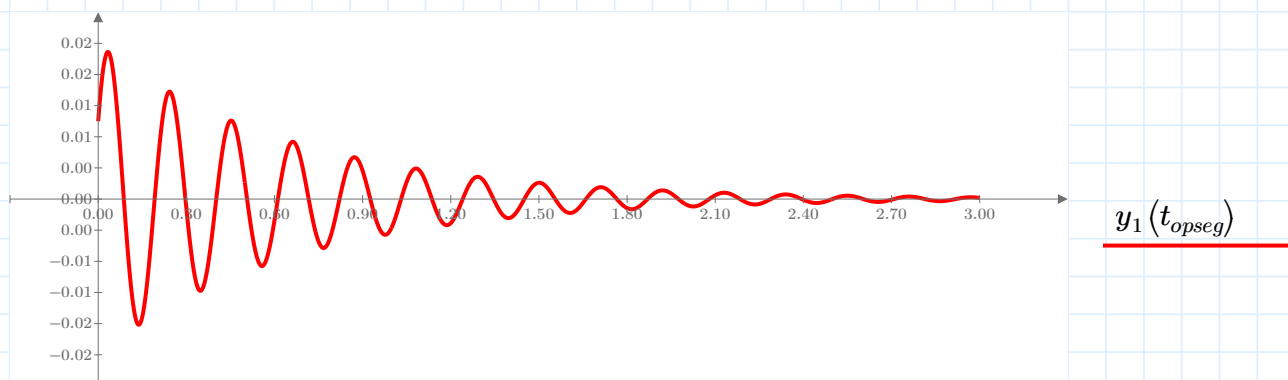
$$\text{Koeficijent prigušenja: } \varepsilon_1 := \frac{c_1}{2 \cdot m} \quad \text{Relativno prigušenje: } \zeta_1 := \frac{\varepsilon_1}{\omega} = 0.05 < 1.0 \quad \text{podkritično prigušenje}$$

$$\text{Kružna frekvencija prigušenih vibracija: } \omega_d := \omega \cdot \sqrt{1 - \zeta_1^2} = 29.962$$

$$\text{Period prigušenih vibracija: } T_d := \frac{2 \cdot \pi}{\omega_d} = 0.210$$

$$\text{Jednačina kretanja: } y_1(t) := e^{-\varepsilon_1 \cdot t} \cdot \left(\frac{v_0 + \varepsilon_1 \cdot y_0}{\omega_d} \cdot \sin(\omega_d \cdot t) + y_0 \cdot \cos(\omega_d \cdot t) \right)$$

$$\text{Grafik: } t_{\text{pocetak}} := 0 \quad t_{\text{korak}} := 0.001 \quad t_{\text{kraj}} := 3 \quad t_{\text{opseg}} := t_{\text{pocetak}} : t_{\text{korak}} : t_{\text{kraj}}$$



$$\text{opseg} := \text{vec}(t_{\text{opseg}}) \quad \text{nizVrednosti}_1 := \overline{y_1(\text{opseg})}$$

$$\max(\text{nizVrednosti}_1) = 0.01890 \\ \min(\text{nizVrednosti}_1) = -0.01615$$

$$C := \sqrt{\left(\frac{v_0 + \varepsilon_1 \cdot y_0}{\omega_d} \right)^2 + y_0^2} = 0.01989 \quad \text{Fazna razlika: } \alpha := \text{atan}\left(\frac{y_0 \cdot \omega_d}{v_0 + \varepsilon_1 \cdot y_0} \right) = 0.52693$$

Trenutak dostizanja prve amplitude: $t_{max_1} := -\frac{\alpha}{\omega_d} + \frac{\pi}{2 \cdot \omega_d} = 0.03484$

$$y_{max} := C \cdot e^{-\varepsilon_1 \cdot t_{max_1}} \cdot \sin(\omega_d \cdot t_{max_1} + \alpha) = 0.01887$$

$$C \cdot e^{-\varepsilon_1 \cdot t_{max_1}} = 0.01887 \quad \sin(\omega_d \cdot t_{max_1} + \alpha) = 1.000$$

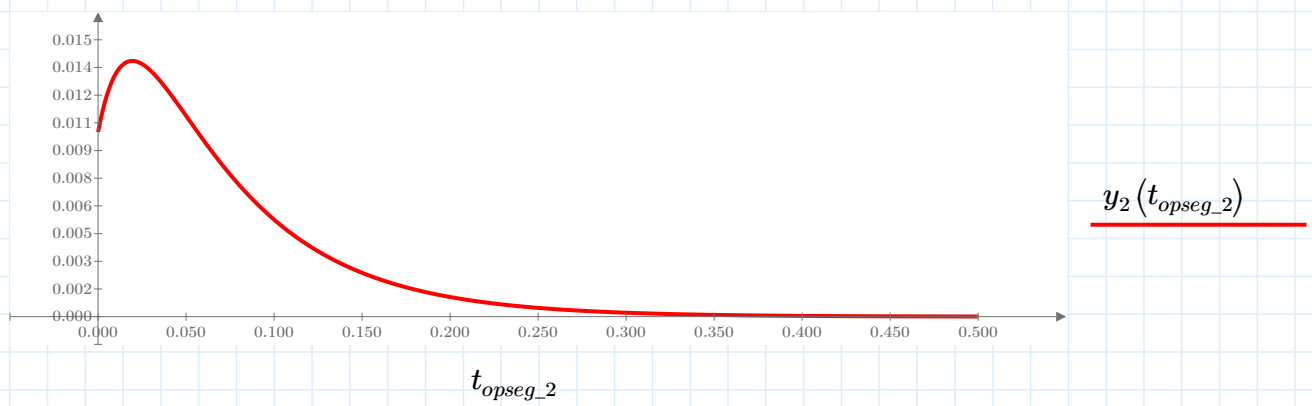
2) Analiza za $c_2 = 1440.00$

$$\varepsilon_2 := \frac{c_2}{2 \cdot m} = 36.00 \quad \zeta_2 := \frac{\varepsilon_2}{\omega} = 1.20 > 1.0 \text{ nadkritično prigušenje}$$

$$A := \frac{v_0 + y_0 \cdot \omega \cdot (\zeta_2 + \sqrt{\zeta_2^2 - 1})}{2 \cdot \omega \cdot \sqrt{\zeta_2^2 - 1}} = 0.0266 \quad B := \frac{-v_0 + y_0 \cdot \omega \cdot (-\zeta_2 + \sqrt{\zeta_2^2 - 1})}{2 \cdot \omega \cdot \sqrt{\zeta_2^2 - 1}} = -0.0166$$

Jednačina kretanja: $y_2(t) := A \cdot e^{(-\zeta_2 + \sqrt{\zeta_2^2 - 1}) \cdot \omega \cdot t} + B \cdot e^{-(\zeta_2 + \sqrt{\zeta_2^2 - 1}) \cdot \omega \cdot t}$

Grafik: $t_{kraj_2} := 0.5 \quad t_{opseg_2} := t_{pocetak}, t_{korak} \dots t_{kraj_2}$



$$opseg_2 := \text{vec}(t_{opseg}) \quad nizVrednosti_2 := \overrightarrow{y_2(opseg_2)}$$

$$\max(nizVrednosti_2) = 0.01385$$

$$\min(nizVrednosti_2) = 0.00000$$

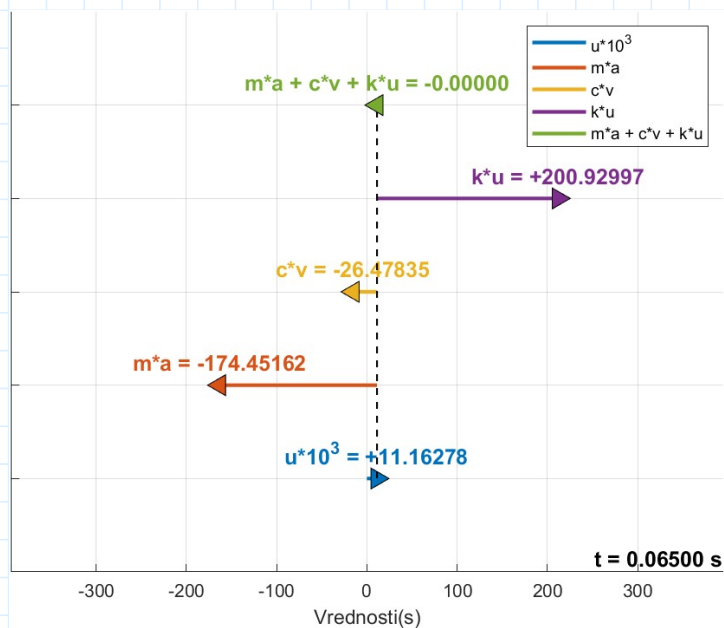
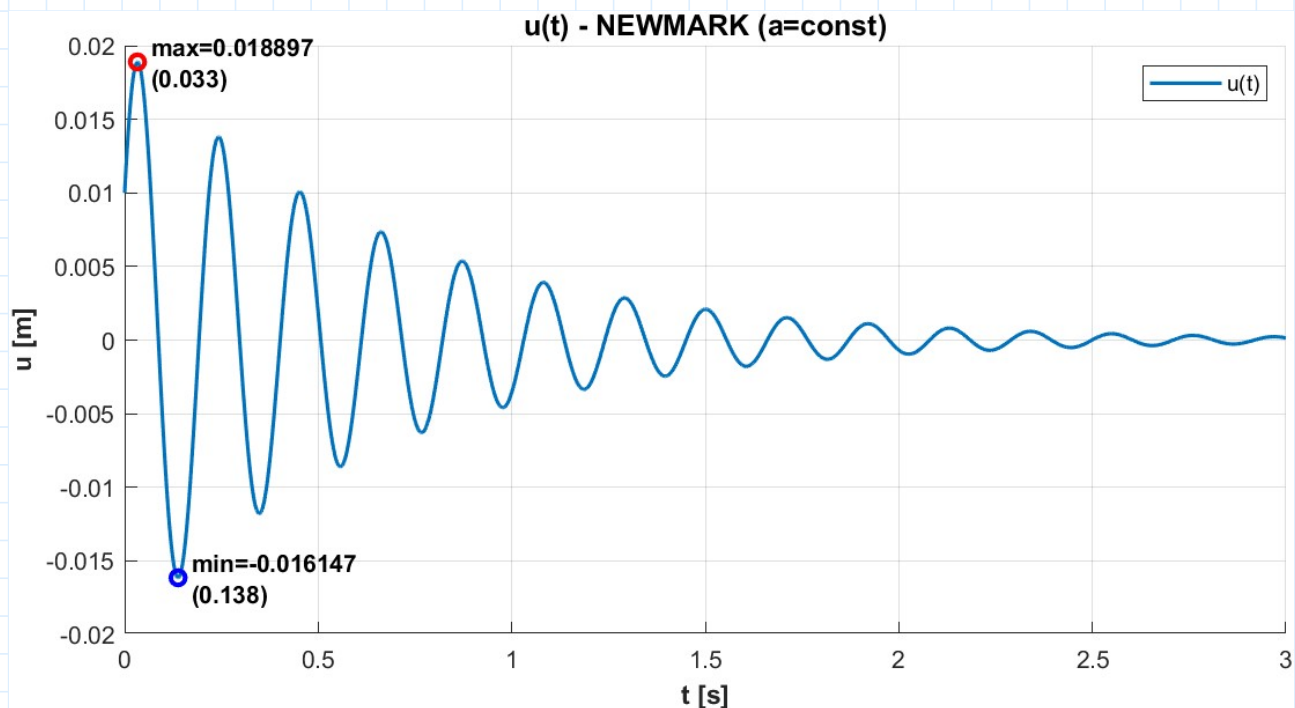
solve, t
assume, 0 < t < 0.05

$$t_{ext} := \frac{d}{dt} y_2(t) = 0 \xrightarrow{\text{float, 6}} 0.0194325$$

$$y_{2_ext} := y_2(t_{ext}) = 0.01386$$

Numerička integracija (Matlab)

1) Analiza za $c_1 = 60.00$ $\zeta_1 = 0.05$



```

m = 20.00000 [t]
k = 18000.00000 [kN/m]
c = 60.00000 [t/s]
ceta = 0.05000
u0 = 0.01000 [m]
v0 = 0.50000 [m/s]

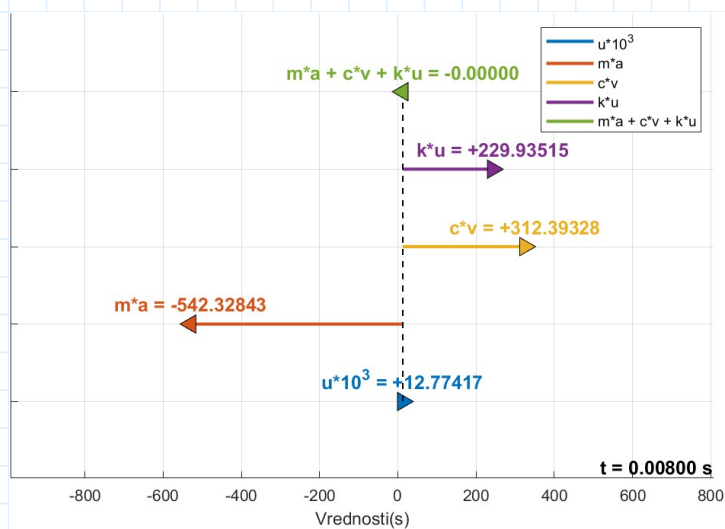
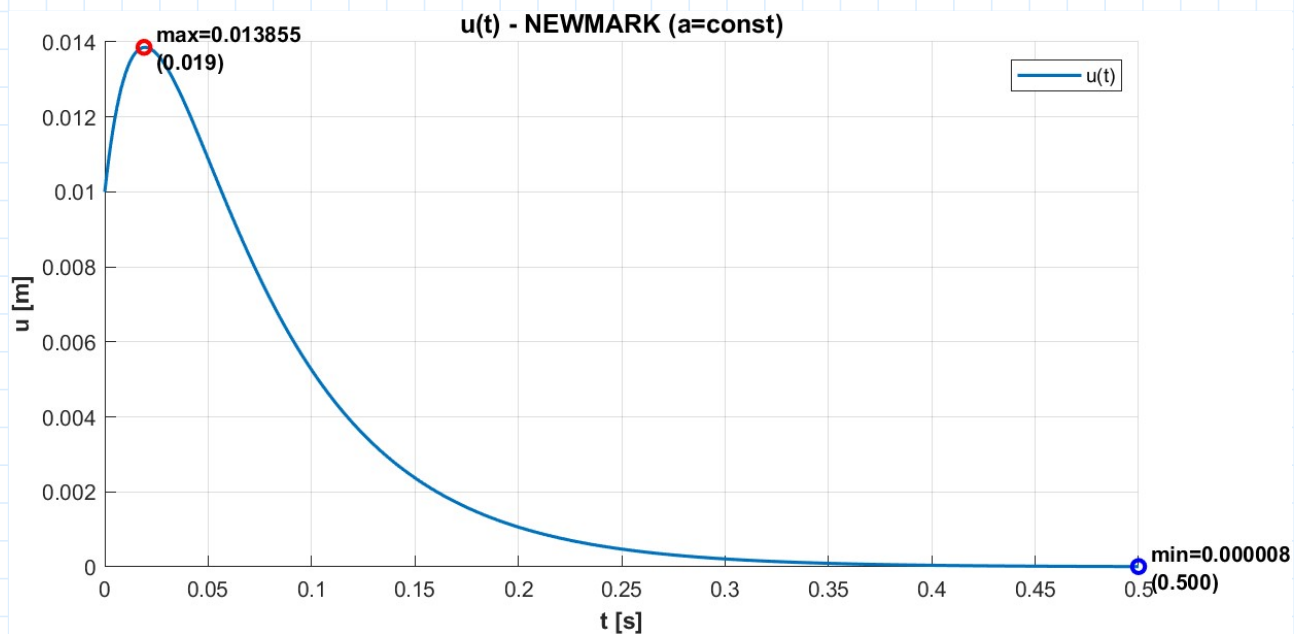
w = 30.00000 [rad/s]
T = 0.20944 [s]
wd = 29.96248 [rad/s]
Td = 0.20970 [s]

umax=0.0188965343 [m] (u t=0.03300 [s])
umin=-0.0161470719 [m] (u t=0.13800 [s])

vmax=0.5000000000 [m/s] (u t=0.00000 [s])
vmin=-0.5253499773 [m/s] (u t=0.08400 [s])

amax=14.6047028379 [m/s/s] (u t=0.13500 [s])
amin=-17.0922734967 [m/s/s] (u t=0.03000 [s])
    
```

2) Analiza za $c_2 = 1440.00$ $\zeta_2 = 1.20$



$m = 20.00000$ [t]
 $k = 18000.00000$ [kN/m]
 $c = 1440.00000$ [t/s]
 $\zeta_2 = 1.20000$
 $u_0 = 0.01000$ [m]
 $v_0 = 0.50000$ [m/s]

$u_{max} = 0.0138553276$ [m] (u t=0.01900[s])
 $u_{min} = 0.0000084882$ [m] (u t=0.50000[s])

$v_{max} = 0.5000000000$ [m/s] (u t=0.00000[s])
 $v_{min} = -0.1348527197$ [m/s] (u t=0.05100[s])

$a_{max} = 1.3125067102$ [m/s/s] (u t=0.08200[s])
 $a_{min} = -45.0000000000$ [m/s/s] (u t=0.00000[s])

Primer 3. Prinudne prigušene vibracije. Harmonijska sila

Sistem sa jednim stepenom slobode.

$$\text{Podaci:} \quad k := 3733.7 \quad m := 4.0775 \quad \zeta := 0.03 \quad F_0 := 15 \quad p := 26.18$$

$$\omega := \sqrt{\frac{k}{m}} = 30.26 \quad T := \frac{2 \cdot \pi}{\omega} = 0.2076 \quad \omega_d := \omega \cdot \sqrt{1 - \zeta^2} = 30.25 \quad T_d := \frac{2 \cdot \pi}{\omega_d} = 0.2077$$

$$\eta := \frac{p}{\omega} = 0.87 \quad \frac{F_0}{k} = 0.0040$$

$$\lambda := \frac{1}{\sqrt{(1 - \eta^2)^2 + 4 \cdot \zeta^2 \cdot \eta^2}} = 3.894 \quad N_p := \frac{F_0}{k} \cdot \lambda = 0.0156 \quad F(t) := F_0 \cdot \sin(p \cdot t)$$

$$\varphi_r := \operatorname{atan}\left(\frac{2 \cdot \zeta \cdot \eta}{1 - \eta^2}\right) = 0.2035 > 0 \quad \varphi := \varphi_r$$

$$y_{\text{transient_a}}(t, y_0, v_0) := e^{-\zeta \cdot \omega \cdot t} \cdot \left(\frac{v_0 + \zeta \cdot \omega \cdot y_0}{\omega_d} \cdot \sin(\omega_d \cdot t) + y_0 \cdot \cos(\omega_d \cdot t) \right)$$

$$y_{\text{transient_b}}(t) := e^{-\zeta \cdot \omega \cdot t} \cdot \left(N_p \cdot \sin(-\varphi) \cos(\omega_d \cdot t) + \frac{N_p}{\omega_d} \cdot (\zeta \cdot \omega \cdot \sin(-\varphi) + p \cdot \cos(-\varphi)) \cdot \sin(\omega_d \cdot t) \right)$$

$$y_{\text{transient_ukupno}}(t, y_0, v_0) := y_{\text{transient_a}}(t, y_0, v_0) - y_{\text{transient_b}}(t)$$

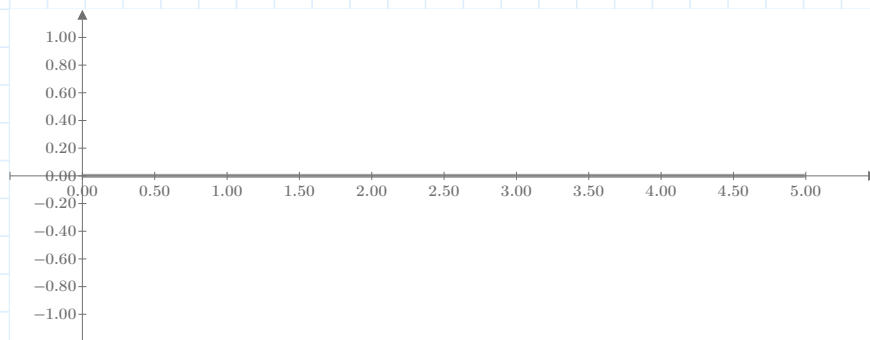
$$y_{\text{steady_state_a}}(t) := N_p \cdot \sin(p \cdot t - \varphi)$$

$$y_{\text{steady_state_b}}(t) := \frac{F_0}{k} \cdot \frac{1}{(1 - \eta^2)^2 + 4 \cdot \zeta^2 \cdot \eta^2} \cdot ((1 - \eta^2) \cdot \sin(p \cdot t) - 2 \cdot \zeta \cdot \eta \cdot \cos(p \cdot t))$$

1) Analiza za: $y_{0_1} := 0$ $v_{0_1} := 0$

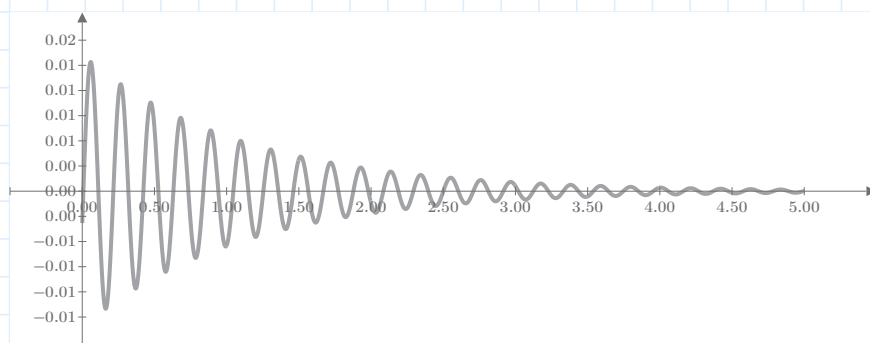
$$y_{\text{transient_a_1}}(t) := y_{\text{transient_a}}(t, y_{0_1}, v_{0_1}) \quad y_{\text{transient_ukupno_1}}(t) := y_{\text{transient_ukupno}}(t, y_{0_1}, v_{0_1})$$

Grafik: $t_{\text{pocetak}} := 0$ $t_{\text{korak}} := 0.001$ $t_{\text{kraj}} := 5$ $t_{\text{opseg}} := t_{\text{pocetak}}, t_{\text{korak}} \cdot t_{\text{kraj}}$



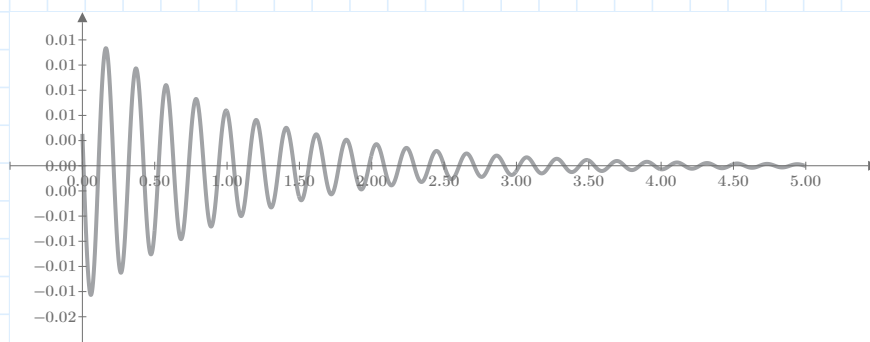
$y_{\text{transient_a_1}}(t_{\text{opseg}})$

t_{opseg}



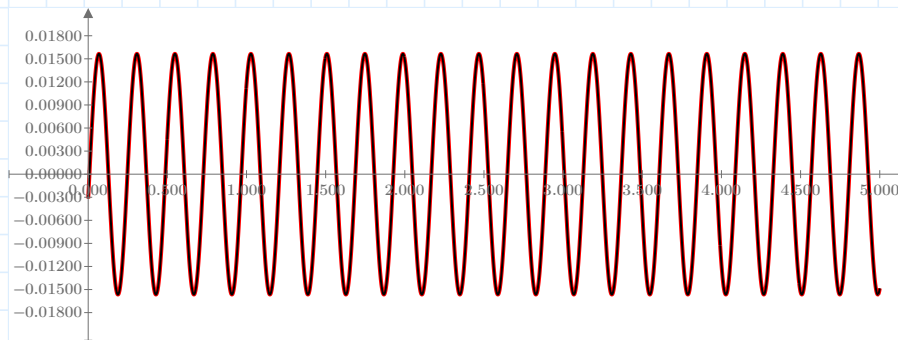
$y_{\text{transient_b}}(t_{\text{opseg}})$

t_{opseg}



$y_{\text{transient_ukupno_1}}(t_{\text{opseg}})$

t_{opseg}



$$\underline{y_{steady_state_a}(t_{opseg})}$$

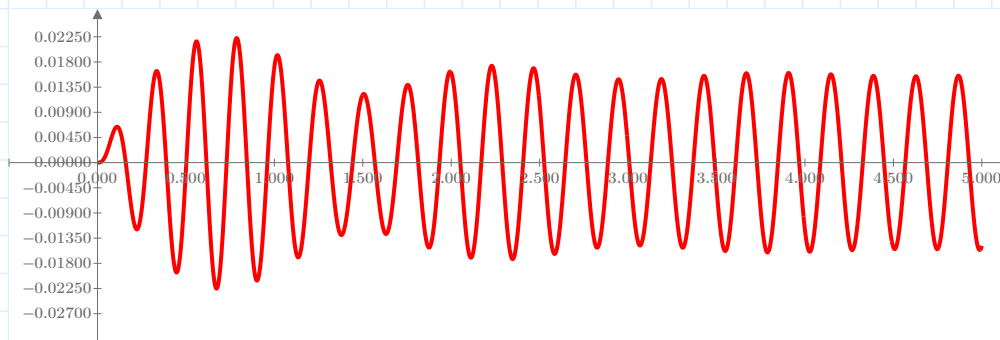
$$\underline{y_{steady_state_b}(t_{opseg})}$$

$$\underline{t_{opseg}}$$

$$opseg := \text{vec}(t_{opseg}) \quad nizVrednosti_1 := \overrightarrow{y_{steady_state_a}(opseg)}$$

$$\max(nizVrednosti_1) = 0.01564 \quad \min(nizVrednosti_1) = -0.01564 \quad N_p = 0.01564$$

$$y_1(t) := y_{transient_ukupno_1}(t) + y_{steady_state_a}(t)$$



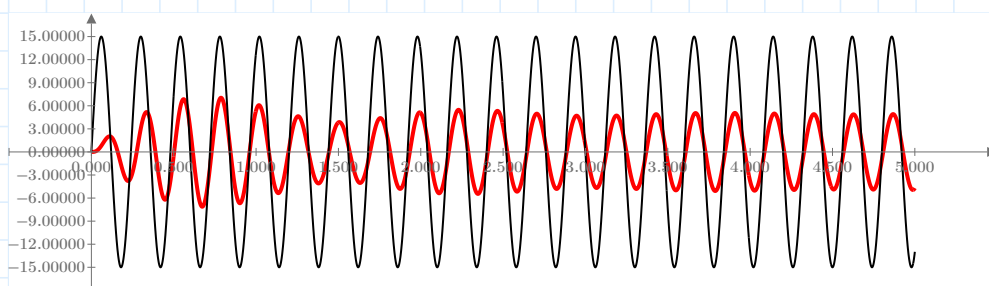
$$\underline{y_1(t_{opseg})}$$

$$\underline{t_{opseg}}$$

$$nizVrednosti_2 := \overrightarrow{y_1(opseg)}$$

$$\max(nizVrednosti_2) = 0.02227 \quad \min(nizVrednosti_2) = -0.02252 \quad N_p = 0.01564$$

Odnos smeri sile i pomeranja



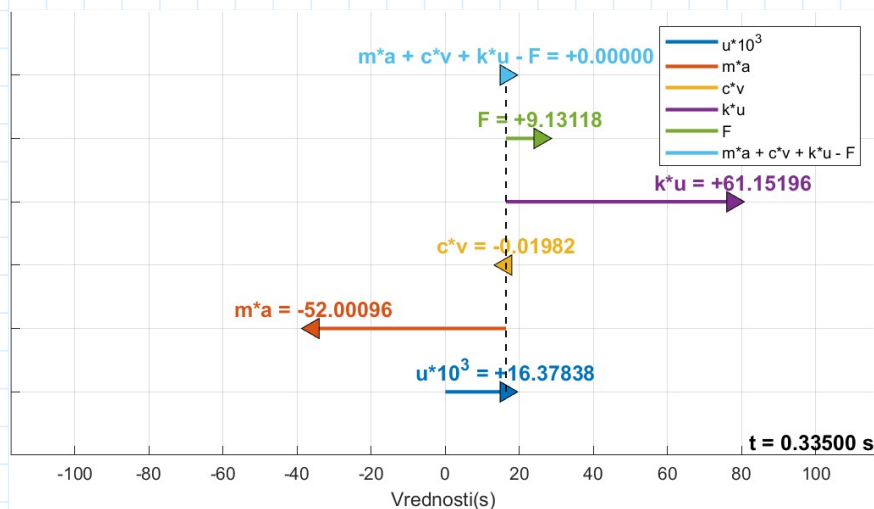
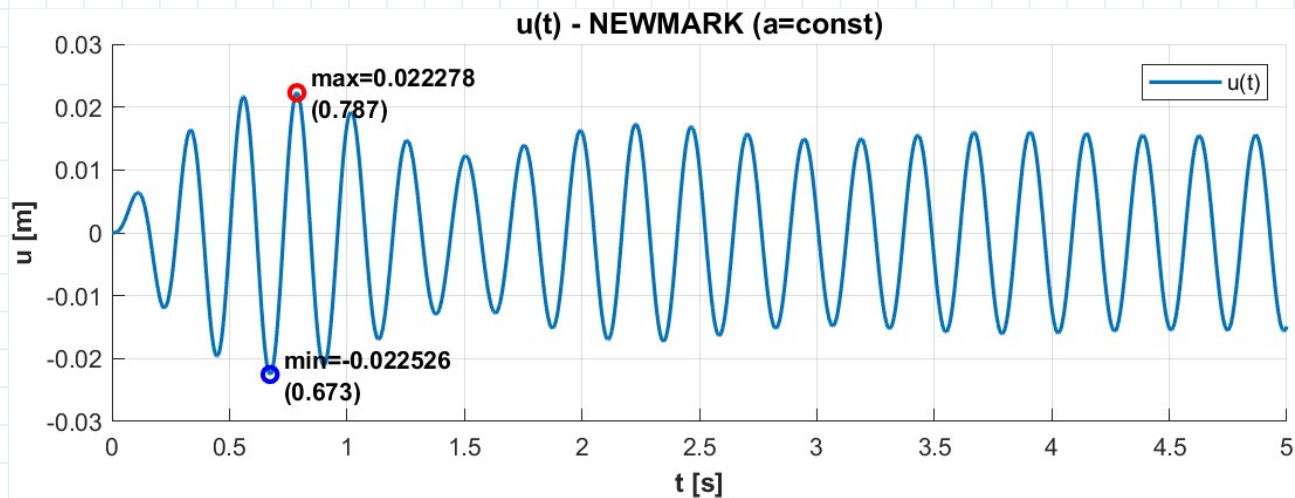
$$\underline{10^{2.5} \cdot y_1(t_{opseg})}$$

$$\underline{F(t_{opseg})}$$

$$\underline{t_{opseg}}$$

Numerička integracija (Matlab)

1) Analiza za: $y_{0,1} = 0.000$ $v_{0,1} = 0.000$



```
m = 4.07750 [t]
k = 3733.70000 [kN/m]
ceta = 0.03000
c = 7.40317 [t/s]
u0 = 0.00000 [m]
v0 = 0.00000 [m/s]
```

```
Amplituda sile F0 = 15.00 [kN]
Kružna frekvencija sile p = 26.18 [rad/s]
Period sile p = 0.24 [s]
```

```
w = 30.26026 [rad/s]
T = 0.20764 [s]
wd = 30.24664 [rad/s]
Td = 0.20773 [s]
```

```
umax=0.0222776999[m] (u t=0.78700[s])
umin=-0.0225255854[m] (u t=0.67300[s])
```

```
vmax=0.6188746957[m/s] (u t=0.73000[s])
vmin=-0.6139956564[m/s] (u t=0.61700[s])
```

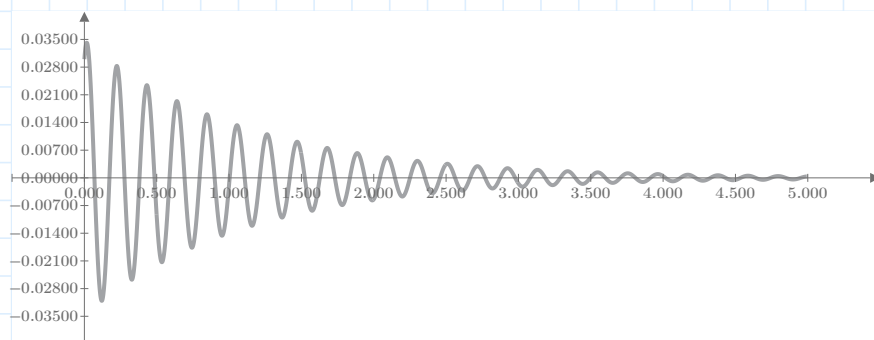
```
amax=17.1656600825[m/s/s] (u t=0.67300[s])
amin=-16.7899626740[m/s/s] (u t=0.78600[s])
```

```
Dinamički koeficijent lambda = 3.89411
Extremno pomeranje od F(t) u_ext = lambda * u_stat_Po = 0.01564 [m]
Extremno pomeranje od F(t) - numerička integracija (posle 3s) u_ext = 0.01606 [m]
```

2) Analiza za: $y_{0_2} := 0.03$ $v_{0_2} := 0.5$

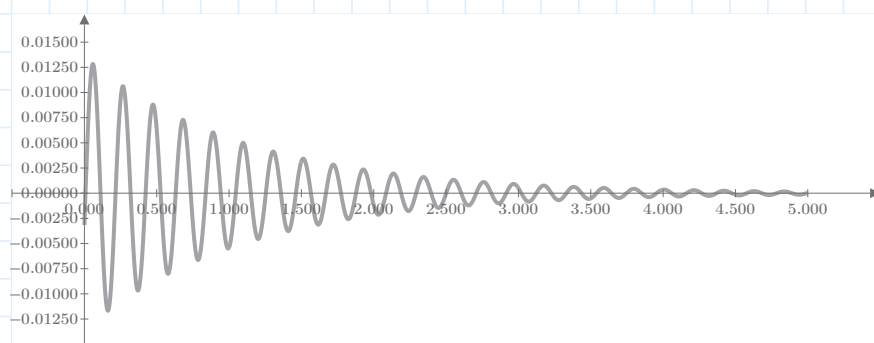
$$y_{\text{transient_a_2}}(t) := y_{\text{transient_a}}(t, y_{0_2}, v_{0_2}) \quad y_{\text{transient_ukupno_2}}(t) := y_{\text{transient_ukupno}}(t, y_{0_2}, v_{0_2})$$

Grafik:



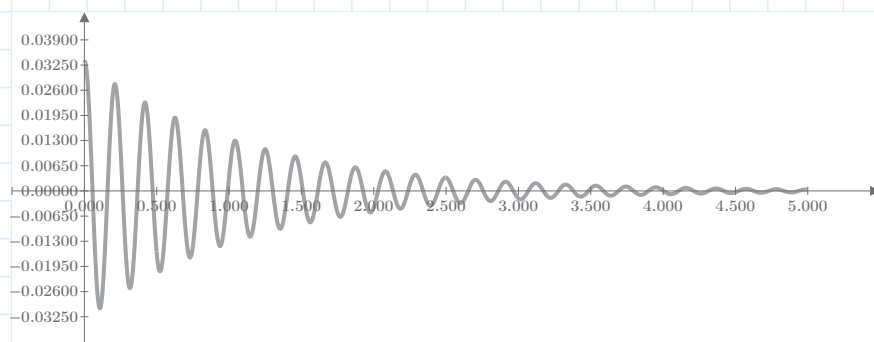
$y_{\text{transient_a_2}}(t_{\text{opseg}})$

t_{opseg}



$y_{\text{transient_b}}(t_{\text{opseg}})$

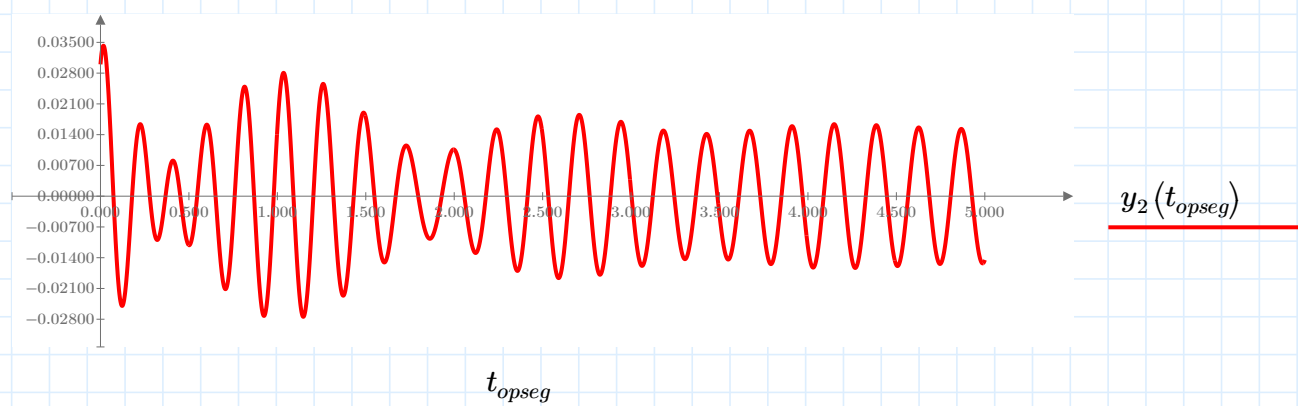
t_{opseg}



$y_{\text{transient_ukupno_2}}(t_{\text{opseg}})$

t_{opseg}

$$y_2(t) := y_{transient_ukupno_2}(t) + y_{steady_state_a}(t)$$



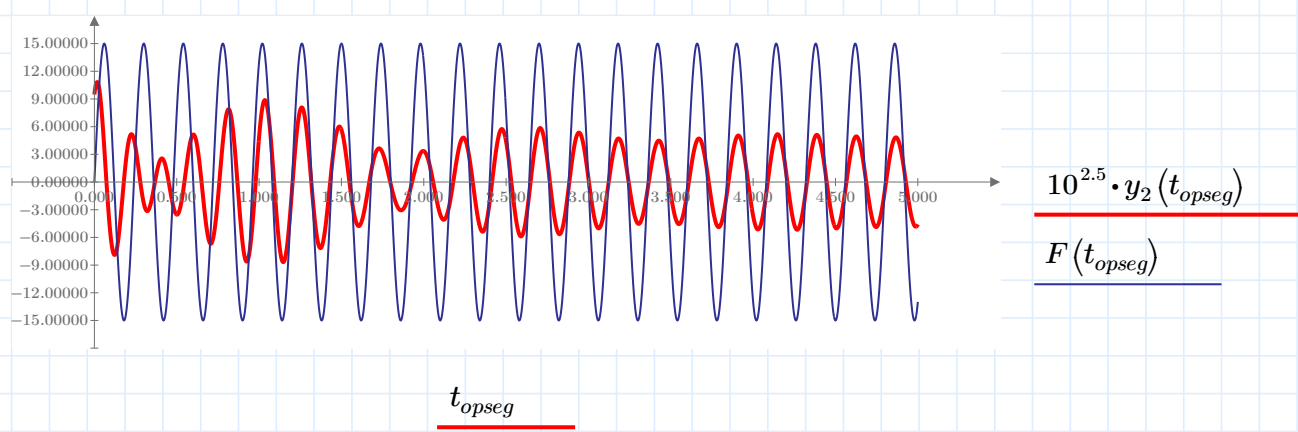
$$nizVrednosti_3 := \overrightarrow{y_2(opseg)}$$

$$\max(nizVrednosti_3) = 0.03424$$

$$\min(nizVrednosti_3) = -0.02744$$

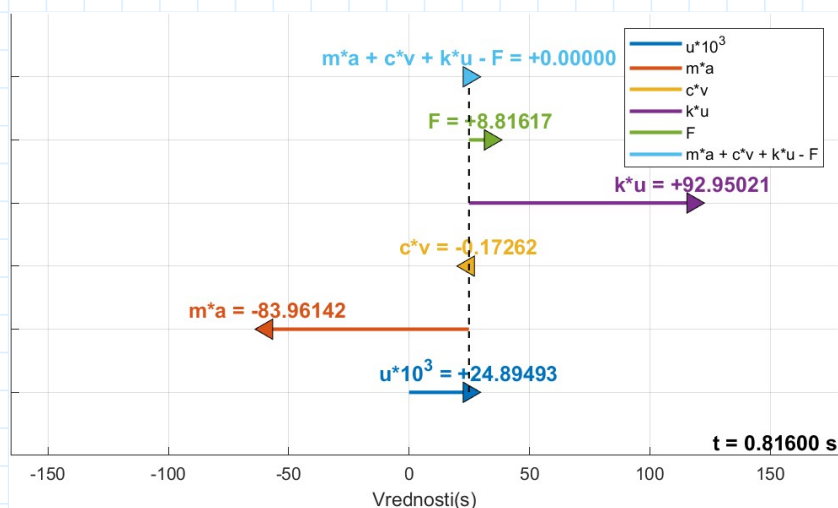
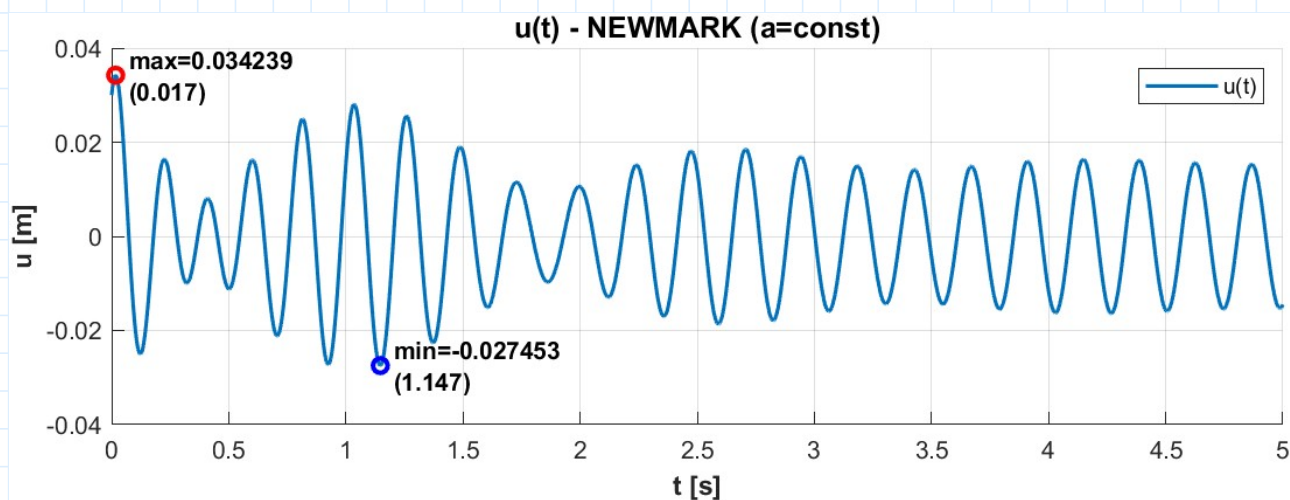
$$N_p = 0.01564$$

Odnos smera sile i pomeranja



Numerička integracija (Matlab)

1) Analiza za: $y_{0,2} = 0.030$ $v_{0,2} = 0.500$



```
m = 4.07750 [t]
k = 3733.70000 [kN/m]
ceta = 0.03000
c = 7.40317 [t/s]
u0 = 0.03000 [m]
v0 = 0.50000 [m/s]
```

```
Amplituda sile F0 = 15.00 [kN]
Kružna frekvencija sile p = 26.18 [rad/s]
Period sile p = 0.24 [s]
```

```
w = 30.26026 [rad/s]
T = 0.20764 [s]
wD = 30.24664 [rad/s]
TD = 0.20773 [s]
```

```
umax=0.0342392687[m] (u t=0.01700[s])
umin=-0.0274526294[m] (u t=1.14700[s])
```

```
vmax=0.7856710125[m/s] (u t=0.98000[s])
vmin=-0.8834693564[m/s] (u t=0.06700[s])
```

```
amax=23.0798881507[m/s/s] (u t=0.11600[s])
amin=-30.1596827193[m/s/s] (u t=0.01100[s])
```

```
Dinamički koeficijent lambda = 3.89411
Extremno pomeranje od F(t) u_ext = lambda * u_stat_Po = 0.01564[m]
Extremno pomeranje od F(t) - numericka integracija (posle 3s) u_ext = 0.01638[m]
```

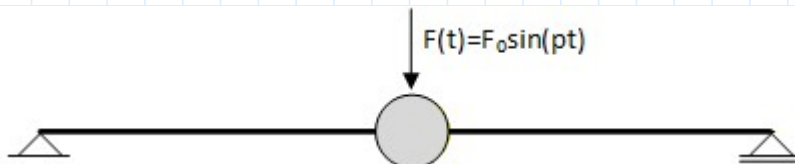
Primer 4. Prinudne prigušene vibracije. Harmonijska sila

Na sredini proste grede nalazi se mašina mase 2000 kg pri čijem radu se javlja harmonijska sila. Broj obrtaja mašine je 560 u minuti. Amplituda prinudne sile mašine je jednaka njenoj petini težine. Homogeni početni uslovi. Analizirati dejstvo harmonijske sile u vertikalnom pravcu. Odrediti ekstremne vrednosti za ugib, moment savijanja i normalni napon.

Podaci: $E=210 \text{ GPa}$, $L=5 \text{ m}$, $\zeta=0,03$, poprečni presek IPE 270 ($A=45,9 \text{ cm}^2$, $I=5790 \text{ cm}^4$, $W=429 \text{ cm}^3$, $M_{grede}=36,1 \text{ kg/m}$).

Dinamički model

Masa modela je koncentrisana u čvor u sredini raspona, tj. na mestu gde se nalazi mašina. Ukupna masa je zbir mase mašine i polovine mase grede. Jedan stepen slobode pomeranja mase u vertikalnom pravcu.



$$M_{grede} := 36.1 \quad M_{masine} := 2000.0 \quad L := 5.0 \quad Em := 210 \cdot 10^6 \quad I := 5790 \cdot 10^{-8}$$

$$m := \left(M_{grede} \cdot \frac{L}{2} + M_{masine} \right) \cdot 10^{-3} = 2.09 \quad k := \frac{48 \cdot Em \cdot I}{L^3} = 4669.06 \quad \omega := \sqrt{\frac{k}{m}} = 47.26$$

$$T := \frac{2 \cdot \pi}{\omega} = 0.1329 \quad \zeta := 0.03 \quad \omega_d := \omega \cdot \sqrt{1 - \zeta^2} = 47.24 \quad T_d := \frac{2 \cdot \pi}{\omega_d} = 0.1330$$

Analiza ustaljenog odgovora

$$n := 560 \quad p := n \cdot \frac{2 \cdot \pi}{60} = 58.64 \quad \eta := \frac{p}{\omega} = 1.24 \quad \lambda_r(\eta, \zeta) := \frac{1}{\sqrt{(1 - \eta^2)^2 + 4 \cdot \zeta^2 \cdot \eta^2}}$$

$$\lambda := \lambda_r(\eta, \zeta) = 1.84 \quad F_0 := \frac{1}{5} \cdot M_{masine} \cdot 9.81 \cdot 10^{-3} = 3.924$$

- Ekstremne vrednosti - ugib

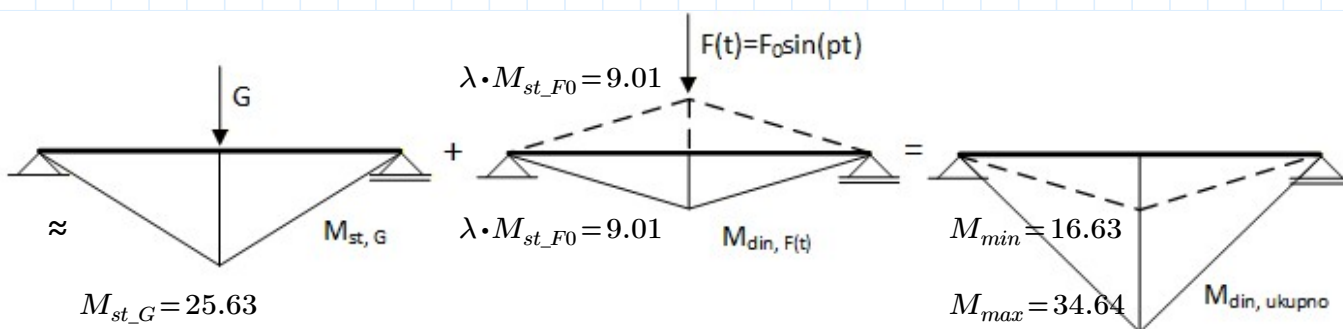
$$G := m \cdot 9.81 = 20.51 \quad y_{st_G} := \frac{G}{k} = 0.00439 \quad y_{st_F0} := \frac{F_0}{k} = 0.00084 \quad y_{din_F} := \lambda \cdot y_{st_F0} = 0.00154$$

$$y_{max} := y_{st_G} + y_{din_F} = 0.00593 \quad y_{min} := y_{st_G} - y_{din_F} = 0.00285$$

- Ekstremne vrednosti - moment savijanja

$$M_{st_G} := \frac{G \cdot L}{4} = 25.63 \quad M_{st_F0} := \frac{F_0 \cdot L}{4} = 4.91 \quad M_{din_F} := \lambda \cdot M_{st_F0} = 9.01$$

$$M_{max} := M_{st_G} + M_{din_F} = 34.64 \quad M_{min} := M_{st_G} - M_{din_F} = 16.63$$



- Ekstremne vrednosti - normalni napon

$W := 429$

1) u trenutku dostizanja Mmax

$$\sigma_{max_1} := \frac{M_{max} \cdot 10^2}{W} = 8.07$$

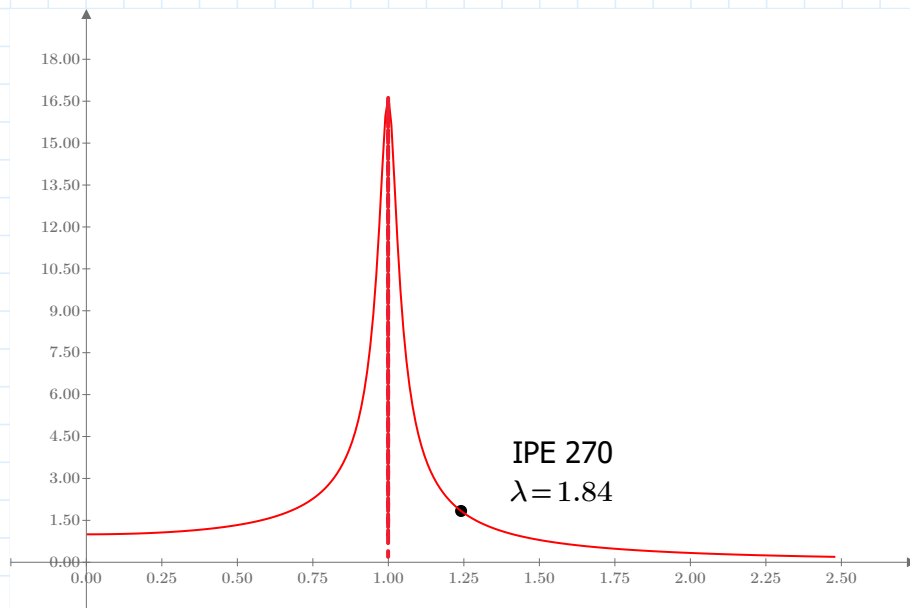
$$\sigma_{min_1} := -\frac{M_{max} \cdot 10^2}{W} = -8.07$$

2) u trenutku dostizanja Mmin

$$\sigma_{max_2} := \frac{M_{min} \cdot 10^2}{W} = 3.88$$

$$\sigma_{min_2} := -\frac{M_{min} \cdot 10^2}{W} = -3.88$$

$$opseg_ \eta := 0, 0.01 \dots 2 \cdot \eta$$



$$\eta \quad opseg_ \eta \quad \sqrt{1 - 2 \cdot \zeta^2}$$

Analiza pomeranja samo usled harmonijske sile

$$\varphi_r := \text{atan}\left(\frac{2 \cdot \zeta \cdot \eta}{1 - \eta^2}\right) = -0.1371 < 0 \quad \varphi := \varphi_r + \pi \quad N_p := \lambda \cdot \frac{F_0}{k} = 0.00154 \quad F(t) := F_0 \cdot \sin(p \cdot t)$$

$$y_{transient_a}(t, y_0, v_0) := e^{-\zeta \cdot \omega \cdot t} \cdot \left(\frac{v_0 + \zeta \cdot \omega \cdot y_0}{\omega_d} \cdot \sin(\omega_d \cdot t) + y_0 \cdot \cos(\omega_d \cdot t) \right)$$

$$y_{transient_b}(t) := e^{-\zeta \cdot \omega \cdot t} \cdot \left(N_p \cdot \sin(-\varphi) \cos(\omega_d \cdot t) + \frac{N_p}{\omega_d} \cdot (\zeta \cdot \omega \cdot \sin(-\varphi) + p \cdot \cos(-\varphi)) \cdot \sin(\omega_d \cdot t) \right)$$

$$y_{transient_ukupno}(t, y_0, v_0) := y_{transient_a}(t, y_0, v_0) - y_{transient_b}(t)$$

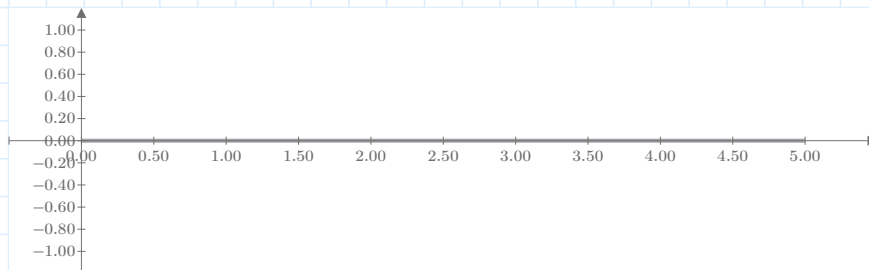
$$y_{steady_state_a}(t) := N_p \cdot \sin(p \cdot t - \varphi)$$

$$y_{steady_state_b}(t) := \frac{F_0}{k} \cdot \frac{1}{(1 - \eta^2)^2 + 4 \cdot \zeta^2 \cdot \eta^2} \cdot ((1 - \eta^2) \cdot \sin(p \cdot t) - 2 \cdot \zeta \cdot \eta \cdot \cos(p \cdot t))$$

$$y_0 := 0 \quad v_0 := 0$$

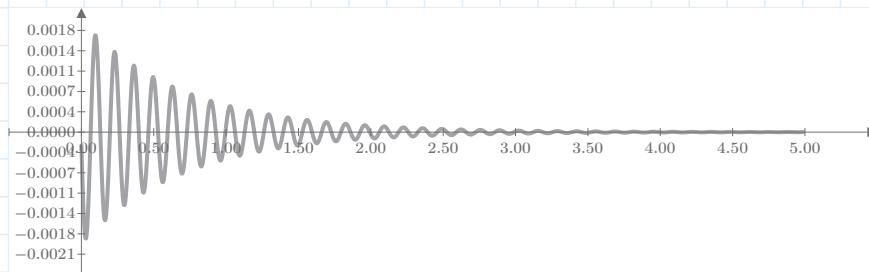
$$y_{\text{transient_a_1}}(t) := y_{\text{transient_a}}(t, y_0, v_0) \quad y_{\text{transient_ukupno_1}}(t) := y_{\text{transient_ukupno}}(t, y_0, v_0)$$

$$\text{Grafik: } t_{\text{pocetak}} := 0 \quad t_{\text{korak}} := 0.001 \quad t_{\text{kraj}} := 5 \quad t_{\text{opseg}} := t_{\text{pocetak}}, t_{\text{korak}} \cdot t_{\text{kraj}}$$



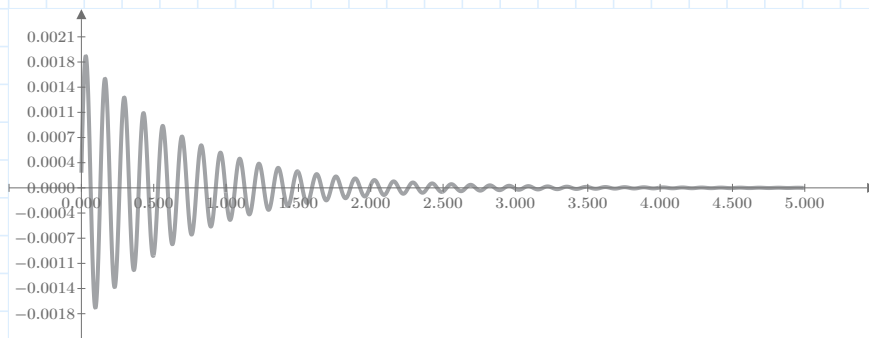
$$y_{\text{transient_a_1}}(t_{\text{opseg}})$$

$$t_{\text{opseg}}$$



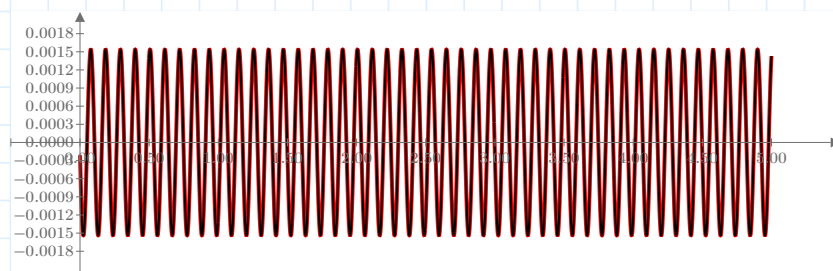
$$y_{\text{transient_b}}(t_{\text{opseg}})$$

$$t_{\text{opseg}}$$



$$y_{\text{transient_ukupno_1}}(t_{\text{opseg}})$$

$$t_{\text{opseg}}$$



$$y_{\text{steady_state_a}}(t_{\text{opseg}})$$

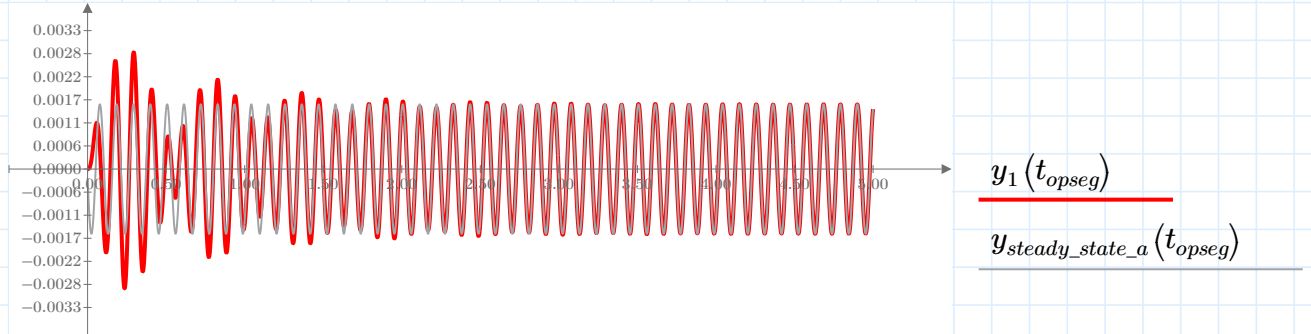
$$y_{\text{steady_state_b}}(t_{\text{opseg}})$$

$$t_{\text{opseg}}$$

$$opseg := \text{vec}(t_{opseg}) \quad nizVrednosti_1 := \overrightarrow{y_{steady_state_a}(opseg)}$$

$$\max(nizVrednosti_1) = 0.00154 \quad \min(nizVrednosti_1) = -0.00154 \quad N_p = 0.00154$$

$$y_1(t) := y_{transient_ukupno_1}(t) + y_{steady_state_a}(t)$$



$$\underline{\underline{t_{opseg}}}$$

$$nizVrednosti_2 := \overrightarrow{y_1(opseg)}$$

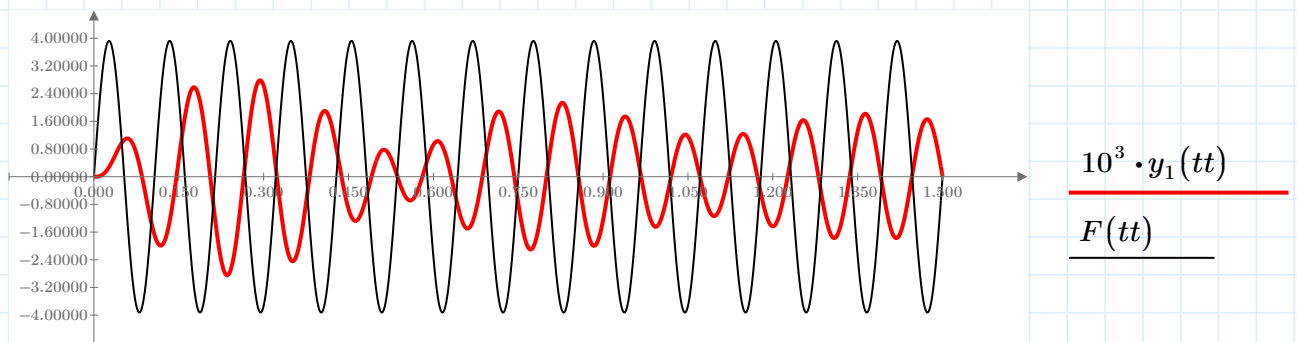
$$\max(nizVrednosti_2) = 0.00279 \quad \min(nizVrednosti_2) = -0.00285 \quad N_p = 0.00154$$

$$\frac{\max(nizVrednosti_2) - N_p}{N_p} \cdot 100 = 80.5 \quad \frac{\text{abs}(\min(nizVrednosti_2)) - N_p}{N_p} \cdot 100 = 84.5$$

Odnos smeru sile i smeru pomeranja

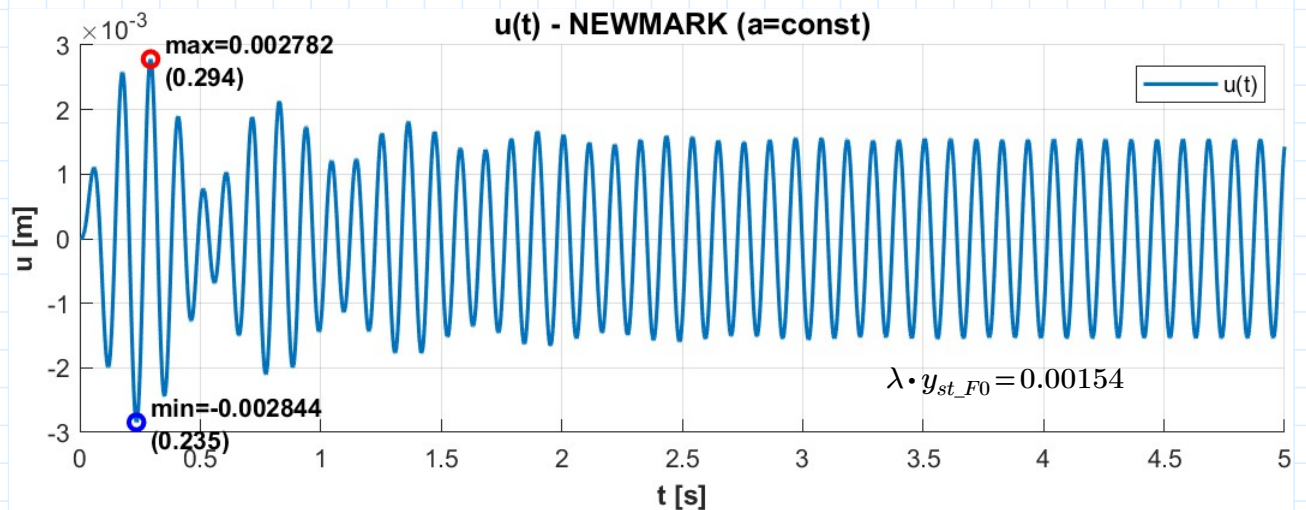
$$tt_{kraj} := 0.3 \cdot t_{kraj} = 1.50$$

$$tt := t_{pocetak}, t_{korak} .. tt_{kraj}$$

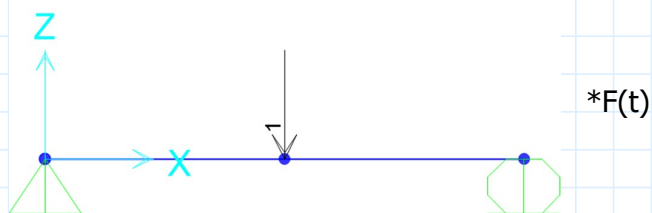


$$\underline{\underline{tt}}$$

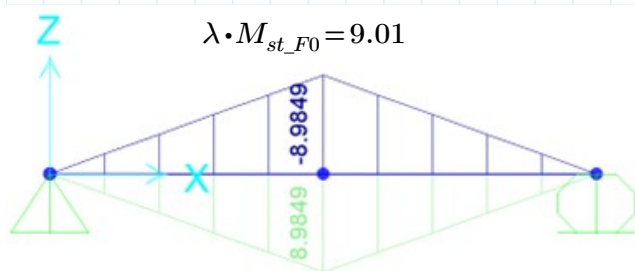
Numerička integracija (Matlab) Pomeranje = transient + steady state (samo od harmonijske sile)



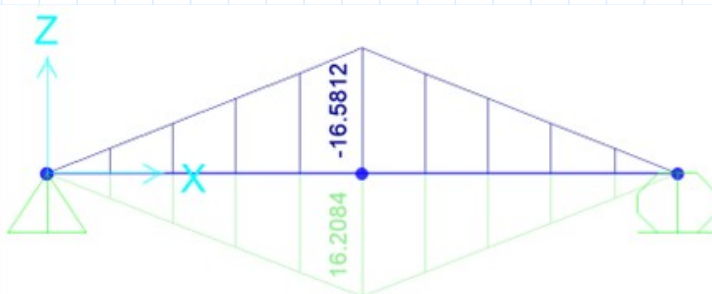
SAP2000 Pomeranje = transient + steady state (samo od harmonijske sile)



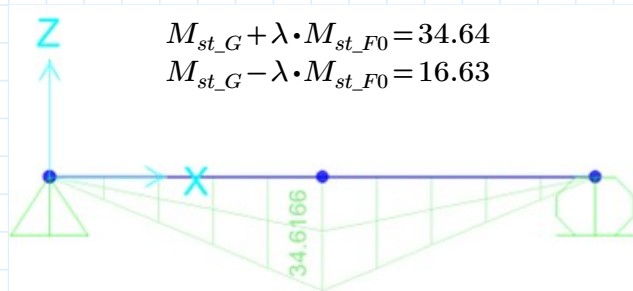
Dijagram M - steady state
(samo od harmonijske sile)



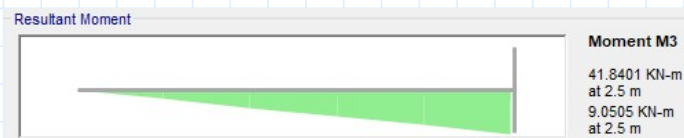
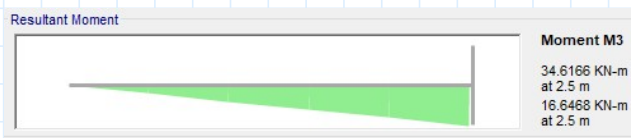
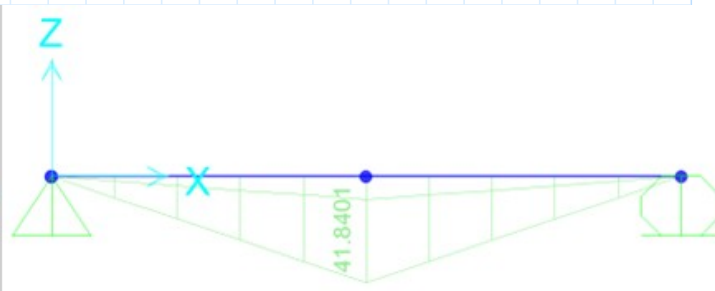
Dijagram M - transient + steady state
(samo od harmonijske sile)



Dijagram M
G+(F(t) steady state)



Dijagram M
G+(F(t) transient + steady state)



KOMENTAR 1.

Postavlja se zahtev da se ekstremi za ugib i normalni napon "smanje".

Rešenje se traži primenom "većeg" poprečnog preseka **IPE 270 -> IPE 300**.

$$M_{grede_2} := 42.2 \quad I_2 := 8356 \cdot 10^{-8}$$

$$m_2 := \left(M_{grede_2} \cdot \frac{L}{2} + M_{masine} \right) \cdot 10^{-3} = 2.11 \quad k_2 := \frac{48 \cdot Em \cdot I_2}{L^3} = 6738.28 \quad \omega_2 := \sqrt{\frac{k_2}{m_2}} = 56.57$$

$$T_2 := \frac{2 \cdot \pi}{\omega_2} = 0.1111 \quad \omega_{d_2} := \omega_2 \cdot \sqrt{1 - \zeta^2} = 56.55 \quad T_{d_2} := \frac{2 \cdot \pi}{\omega_{d_2}} = 0.1111 \quad \begin{array}{l} \text{IPE270} \\ T_d = 0.1330 \end{array}$$

$$\eta_2 := \frac{p}{\omega_2} = 1.04 \quad \lambda_2 := \lambda_r(\eta_2, \zeta) = 10.30 \quad \begin{array}{l} \text{IPE270} \\ \lambda = 1.84 \end{array}$$

Ekstremne vrednosti - ugib

$$G_2 := m_2 \cdot 9.81 = 20.65 \quad y_{st_G_2} := \frac{G_2}{k_2} = 0.00307$$

$$y_{st_F0_2} := \frac{F_0}{k_2} = 0.00058 \quad y_{din_F_2} := \lambda_2 \cdot y_{st_F0_2} = 0.00600$$

IPE 300:

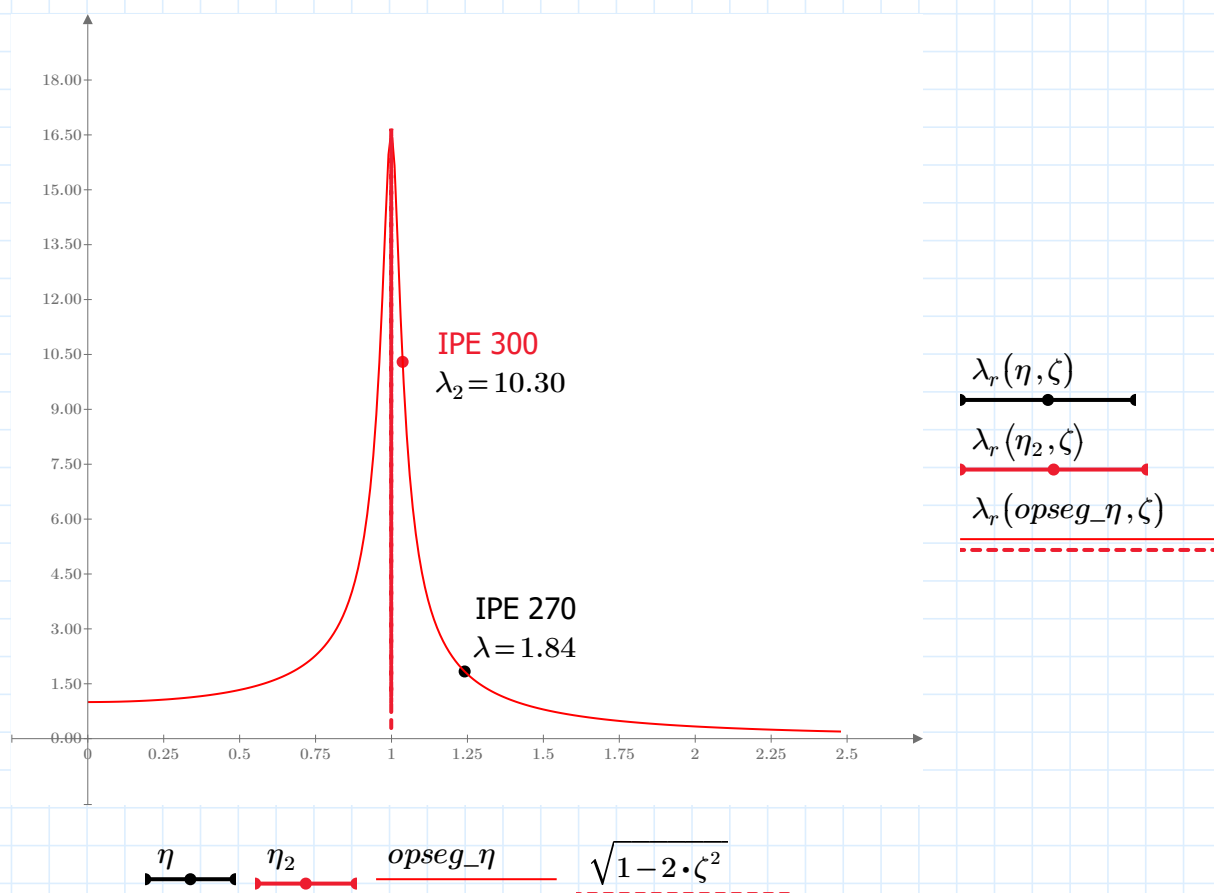
$$y_{max_2} := y_{st_G_2} + y_{din_F_2} = 0.00906$$

$$y_{min_2} := y_{st_G_2} - y_{din_F_2} = -0.00293$$

IPE 270:

$$y_{max} = 0.00593$$

$$y_{min} = 0.00285$$



Ekstremne vrednosti - moment savijanja

$$M_{st_G_2} := \frac{G_2 \cdot L}{4} = 25.82 \quad M_{st_F0} = 4.91 \quad M_{din_F_2} := \lambda_2 \cdot M_{st_F0} = 50.51$$

$$\text{IPE 300: } M_{max_2} := M_{st_G_2} + M_{din_F_2} = 76.33 \quad M_{min_2} := M_{st_G_2} - M_{din_F_2} = -24.69$$

$$\text{IPE 270: } M_{max} = 34.64 \quad M_{min} = 16.63$$

Ekstremne vrednosti - normalni napon

$$\text{IPE 300: } W_2 := 557 \quad \begin{array}{ll} 1) \text{ u trenutku dostizanja Mmax} & 2) \text{ u trenutku dostizanja Mmin} \\ \sigma_{max_1_2} := \frac{M_{max_2} \cdot 10^2}{W_2} = 13.70 & \sigma_{max_2_2} := \frac{M_{min_2} \cdot 10^2}{W_2} = -4.43 \end{array}$$

$$\sigma_{min_1_2} := -\frac{M_{max_2} \cdot 10^2}{W_2} = -13.70 \quad \sigma_{min_2_2} := -\frac{M_{min_2} \cdot 10^2}{W_2} = 4.43$$

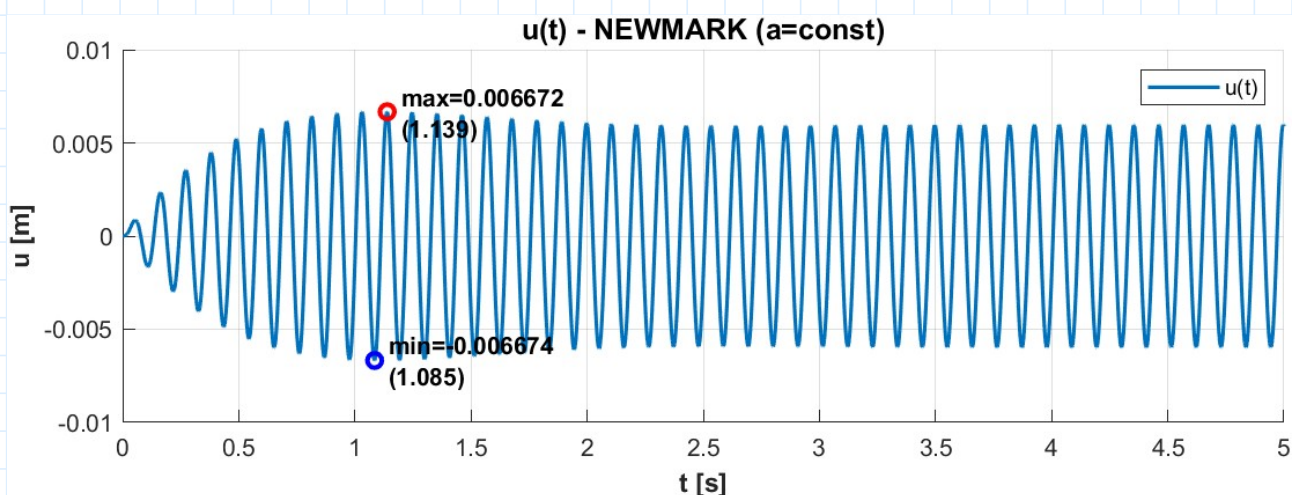
$$\text{IPE 270: } \begin{array}{ll} 1) \text{ u trenutku dostizanja Mmax} & 2) \text{ u trenutku dostizanja Mmin} \end{array}$$

$$\sigma_{max_1} = 8.07 \quad \sigma_{max_2} = 3.88$$

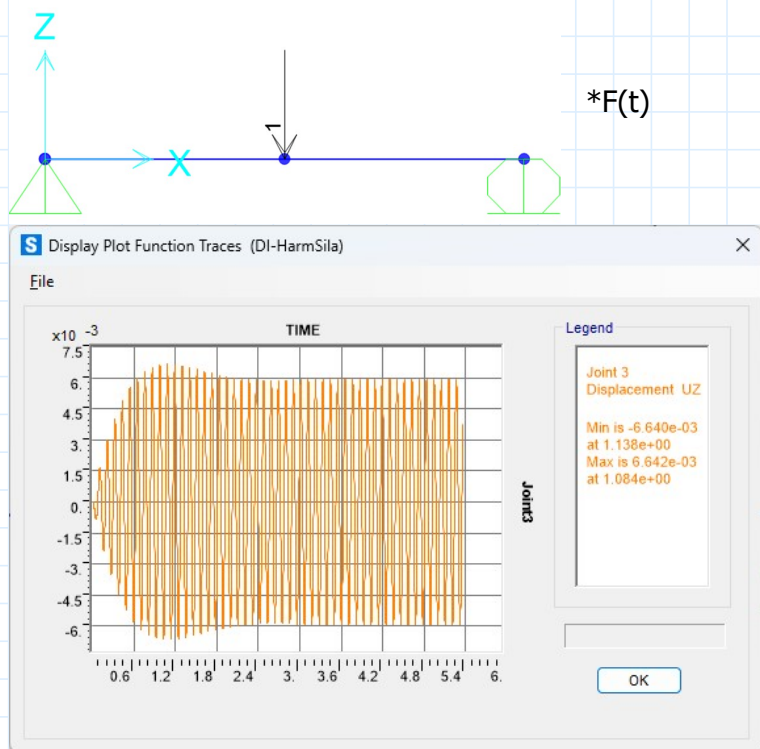
$$\sigma_{min_1} = -8.07 \quad \sigma_{min_2} = -3.88$$

Zaključak: ekstremi za ugib i normalni napon su se "povećali".

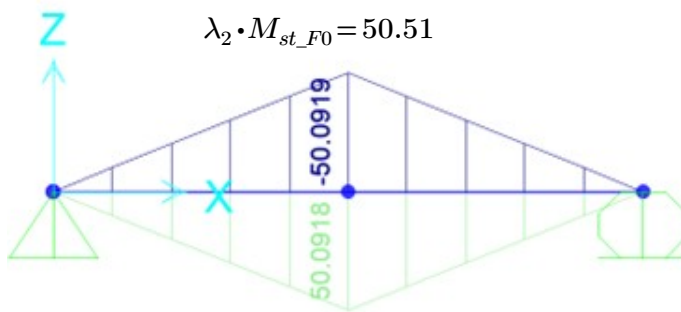
Numerička integracija (Matlab) Pomeranje = transient + steady state (samo od harmonijske sile)



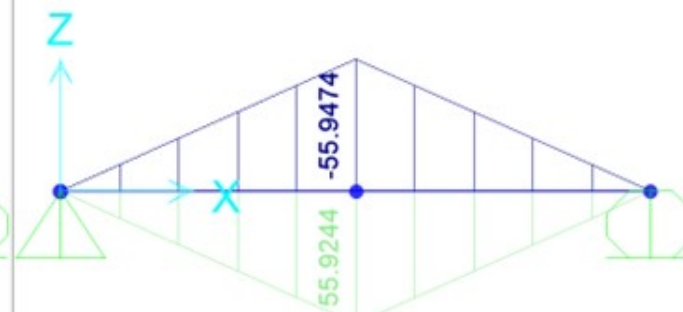
SAP2000 Pomeranje = transient + steady state (samo od harmonijske sile)



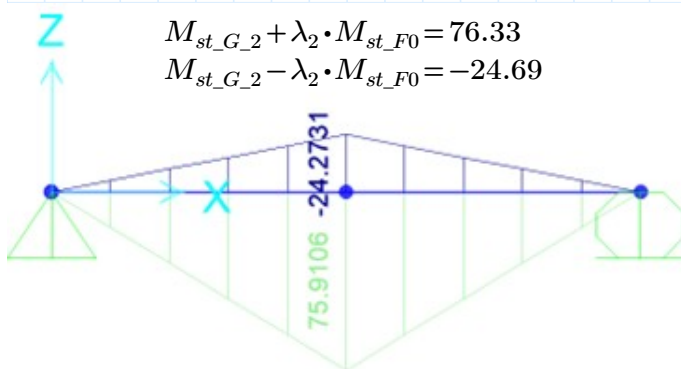
Dijagram M - steady state
(samo od harmonijske sile)



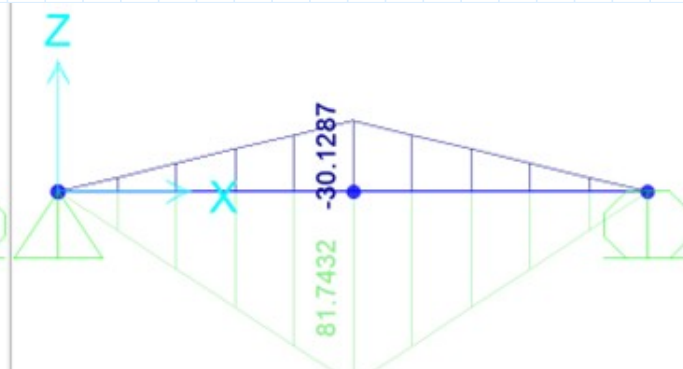
Dijagram M - transient + steady state
(samo od harmonijske sile)



Dijagram M
G+(F(t) steady state)



Dijagram M
G+(F(t) transient + steady state)



IPE 270: $M_{st_G} + \lambda \cdot M_{st_F0} = 34.64$
 $M_{st_G} - \lambda \cdot M_{st_F0} = 16.63$

KOMENTAR 2.

Rešenje se traži primenom "još većeg" poprečnog preseka **IPE 300 -> IPE 400**.

$$M_{grede_3} := 66.3 \quad I_3 := 23130 \cdot 10^{-8}$$

$$m_3 := \left(M_{grede_3} \cdot \frac{L}{2} + M_{masine} \right) \cdot 10^{-3} = 2.17 \quad k_3 := \frac{48 \cdot E m \cdot I_3}{L^3} = 18652.03 \quad \omega_3 := \sqrt{\frac{k_3}{m_3}} = 92.80$$

$$T_3 := \frac{2 \cdot \pi}{\omega_3} = 0.0677 \quad \omega_{d_3} := \omega_3 \cdot \sqrt{1 - \zeta^2} = 92.76 \quad T_{d_3} := \frac{2 \cdot \pi}{\omega_{d_3}} = 0.0677$$

$$\eta_3 := \frac{p}{\omega_3} = 0.63 \quad \lambda_3 := \lambda_r(\eta_3, \zeta) = 1.66 \quad \text{IPE270: } \lambda = 1.84 \quad \text{IPE300: } \lambda_2 = 10.30$$

Ekstremne vrednosti - ugib

$$G_3 := m_3 \cdot 9.81 = 21.25 \quad y_{st_G_3} := \frac{G_3}{k_3} = 0.00114$$

$$y_{st_F0_3} := \frac{F_0}{k_3} = 0.00021 \quad y_{din_F_3} := \lambda_3 \cdot y_{st_F0_3} = 0.0003495$$

IPE 400:

$$y_{max_3} := y_{st_G_3} + y_{din_F_3} = 0.00149$$

$$y_{min_3} := y_{st_G_3} - y_{din_F_3} = 0.00079$$

IPE 300:

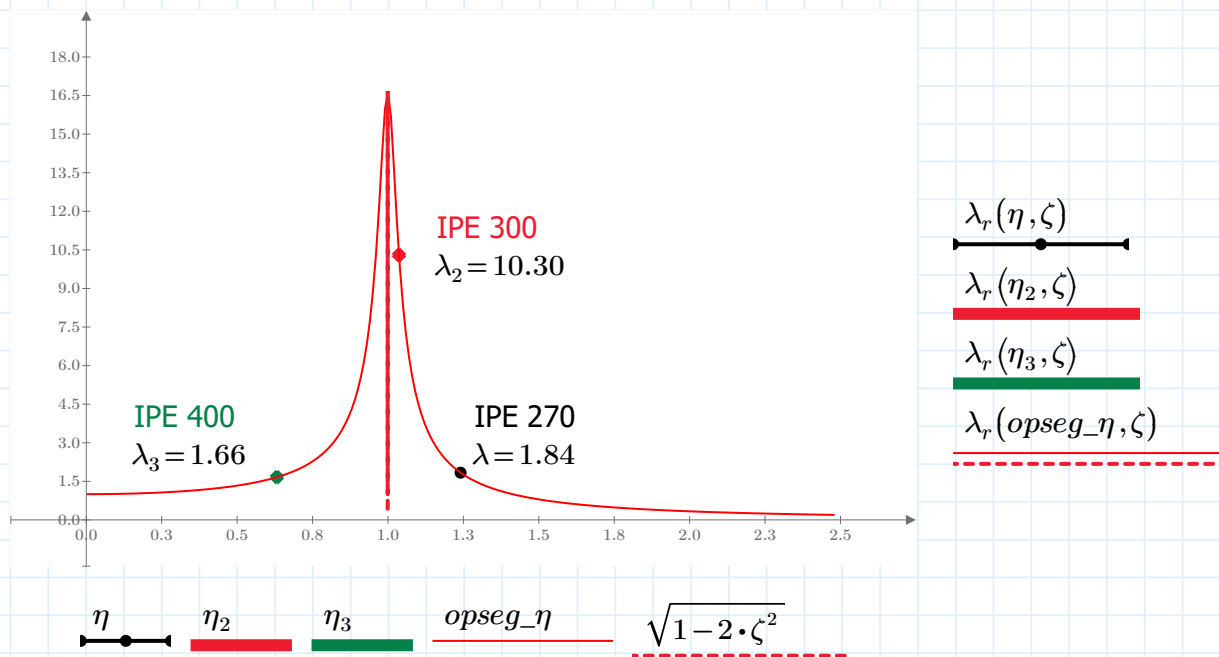
$$y_{max_2} = 0.00906$$

$$y_{min_2} = -0.00293$$

IPE 270:

$$y_{max} = 0.00593$$

$$y_{min} = 0.00285$$



Ekstremne vrednosti - moment savijanja

$$M_{st_G_3} := \frac{G_3 \cdot L}{4} = 26.56 \quad M_{st_F0} = 4.91$$

$$M_{din_F_3} := \lambda_3 \cdot M_{st_F0} = 8.15$$

IPE 400: $M_{max_3} := M_{st_G_3} + M_{din_F_3} = 34.71$

$$M_{min_3} := M_{st_G_3} - M_{din_F_3} = 18.41$$

IPE 300: $M_{max_2} = 76.33$

$$M_{min_2} = -24.69$$

IPE 270: $M_{max} = 34.64$

$$M_{min} = 16.63$$

Ekstremne vrednosti - normalni napon

IPE 400: $W_3 := 1156$ 1) u trenutku dostizanja Mmax

$$\sigma_{max_1_3} := \frac{M_{max_3} \cdot 10^2}{W_3} = 3.00$$

2) u trenutku dostizanja Mmin

$$\sigma_{max_2_3} := \frac{M_{min_3} \cdot 10^2}{W_3} = 1.59$$

$$\sigma_{min_1_3} := -\frac{M_{max_3} \cdot 10^2}{W_3} = -3.00$$

$$\sigma_{min_2_3} := -\frac{M_{min_3} \cdot 10^2}{W_3} = -1.59$$

IPE 300: 1) u trenutku dostizanja Mmax

$$\sigma_{max_1_2} = 13.70$$

2) u trenutku dostizanja Mmin

$$\sigma_{max_2_2} = -4.43$$

$$\sigma_{min_1_2} = -13.70$$

$$\sigma_{min_2_2} = 4.43$$

IPE 270: 1) u trenutku dostizanja Mmax

$$\sigma_{max_1} = 8.07$$

2) u trenutku dostizanja Mmin

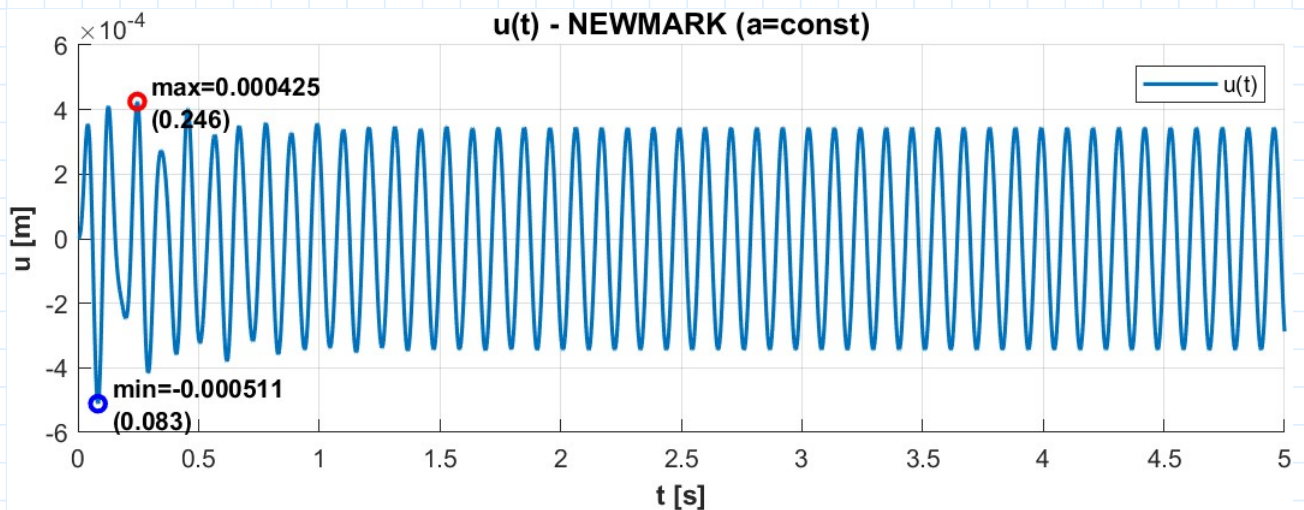
$$\sigma_{max_2} = 3.88$$

$$\sigma_{min_1} = -8.07$$

$$\sigma_{min_2} = -3.88$$

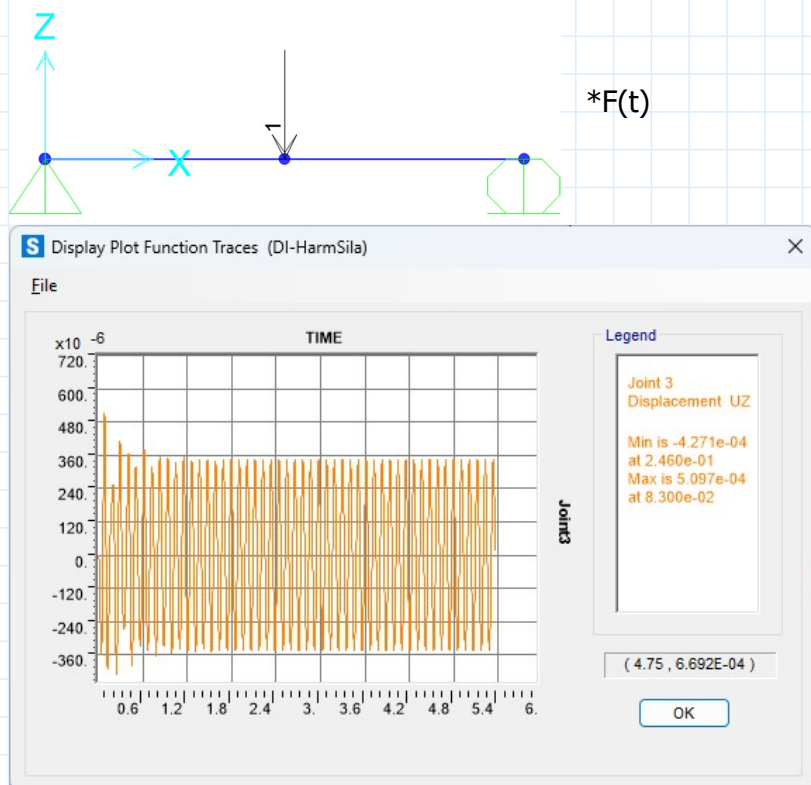
Zaključak: ekstremi za ugib i normalni napon su se "smanjili".

Numerička integracija (Matlab) Pomeranje = transient + steady state (samo od harmonijske sile)



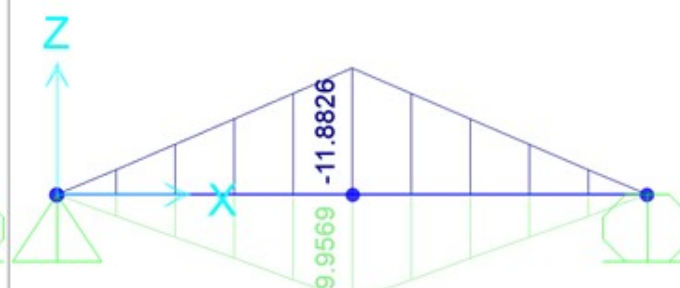
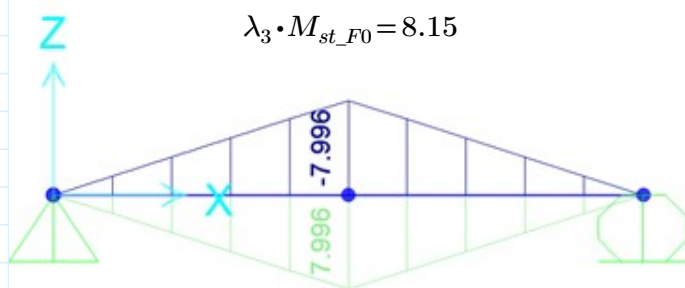
SAP2000

Pomeranje = transient + steady state (samo od harmonijske sile)



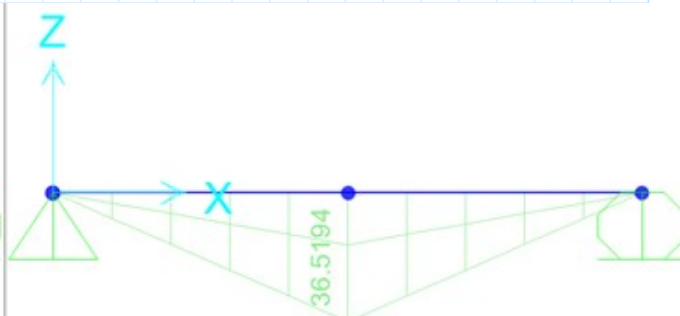
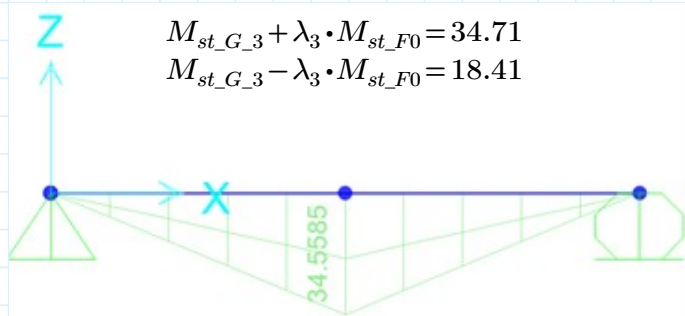
Dijagram M - steady state
(samo od harmonijske sile)

Dijagram M - transient + steady state
(samo od harmonijske sile)



Dijagram M
G+(F(t) steady state)

Dijagram M
G+(F(t) transient + steady state)



IPE 400:	$M_{max_3} = 34.71$	$M_{min_3} = 18.41$
IPE 300:	$M_{max_2} = 76.33$	$M_{min_2} = -24.69$
IPE 270:	$M_{max} = 34.64$	$M_{min} = 16.63$

KOMENTAR 3.

Analizira se ravnomerno povećanje broj obrtaja ekscentrične mase od 0 obrt./min do maksimalnog broja obrtaja $n_{\max} = 560$ obrt./min koje traje $t_{\max}$. Homogeni početni uslovi. Analizira se vertikalna komponenta centrifugalne sile i modeli sa presecima IPE270 i IPE400.

Da bi se analiza uskladila sa harmonijskom silom pri maksimalnom broju obrtaja usvaja se masa m i određuje se ekscentricitet mase iz uslova da se postigne amplituda harmonijske sile pri maksimalnom broju obrtaja

Period sile pri maksimalnom broju obrtaja: $T_{sile} := \frac{2 \cdot \pi}{p} = 0.1071$

Usvaja se: $t_{\max} := 3.0$ $m_e := 0.1$

Ravnomerno promenljiv broj obrtaja (linearna promena broja obrtaja) prouzrokuje linearnu promenu ugaone brzine mase (konstantno ubrzanje mase) u opsegu od 0 obrt./min do $n_{\max} = 560$ obrt./min

Promena broja obrtaja: $n_{\max} := n = 560.0$ $nn(t) := \frac{n_{\max}}{t_{\max}} \cdot t$

Ugaona brzina mase: $pp(t) := \frac{2 \cdot \pi}{60} \cdot \frac{n_{\max}}{t_{\max}} \cdot t$ $p_{\max} := pp(t_{\max}) = 58.643$

Određivanje ekscentriciteta r (centrifugalna sila pri $p_{\max}$):

Maksimalno normalno ubrzanje: $a_{n_max}(r_{rac}) := r_{rac} \cdot p_{\max}^2$
 $r := m_e \cdot a_{n_max}(r_{rac}) = F_0 \xrightarrow{\text{solve}, r_{rac}} 0.011410264418089289924$

Normalno ubrzanje tokom promene broja obrtaja:

$$a_n(t) := r \cdot (pp(t))^2$$

Centrifugalna sila tokom promene broja obrtaja:

$F_c(t) := m_e \cdot a_n(t)$ kontrola ($F_0 = F_{c_max}$): $F_{c_max} := m_e \cdot a_n(t_{\max}) = 3.924$

Promena ugla ϕ (izvod po vremenu ugaone brzine: $d\phi/dt = p$):

$$\phi_1(t) := \int_0^t pp(t) dt \rightarrow 9.773843811168245631 \cdot t^2 \quad \phi_1(t_{\max}) = 87.965$$

ili

$$\phi_2(t) := \frac{2 \cdot \pi}{60} \cdot \frac{n_{\max}}{t_{\max}} \cdot \frac{t^2}{2} \rightarrow 9.7738438111682456308 \cdot t^2 \quad \phi_2(t_{\max}) = 87.965$$

Vertikalna komponenta harmonijske sile tokom promene broja obrtaja:

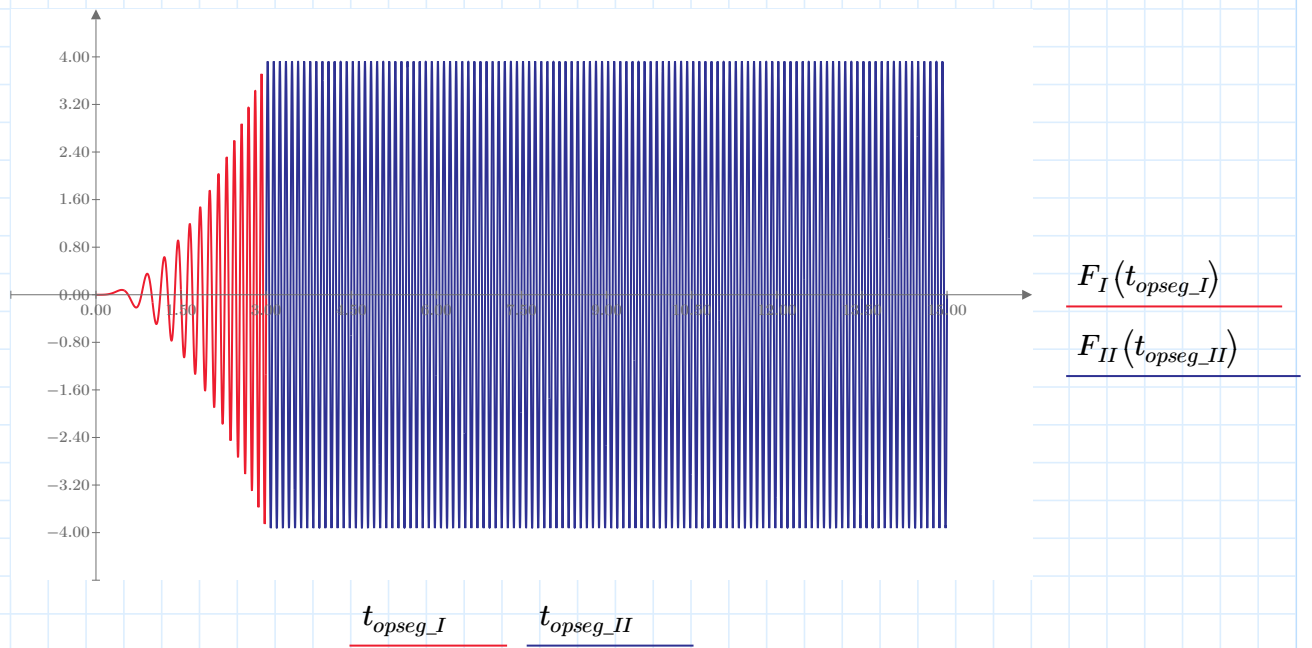
$$F_I(t) := F_c(t) \cdot \sin(\phi_2(t)) \xrightarrow{\text{float}, 10} 0.436 \cdot t^2 \cdot \sin(9.773843811 \cdot t^2)$$

Vertikalna komponenta harmonijske sile tokom maksimalnog broja obrtaja:

$$F_{II}(t) := F_{c_max} \cdot \sin(p_{\max} \cdot (t - t_{\max}) + \phi_2(t_{\max})) \xrightarrow{\text{float}, 10} 3.924 \cdot \sin(58.64306287 \cdot t - 87.9645943)$$

$koefZaKraj := 3$

$t_{opseg_I} := 0, t_{korak} \dots t_{max} \quad t_{opseg_II} := t_{max}, t_{max} + t_{korak} \dots koefZaKraj \cdot t_{kraj}$



$ceoOpseg := t_{pocetak}, t_{korak} \dots koefZaKraj \cdot t_{kraj}$

$opsegVektor := \text{vec}(ceoOpseg) \quad vreme := \text{WRITETEXT}("vreme.txt", opsegVektor)$

$a := \overrightarrow{F_I(\text{vec}(t_{opseg_I}))}$

$b := \overrightarrow{F_{II}(\text{vec}(t_{opseg_II}))}$

$R := \text{WRITETEXT}("R.txt", a)$

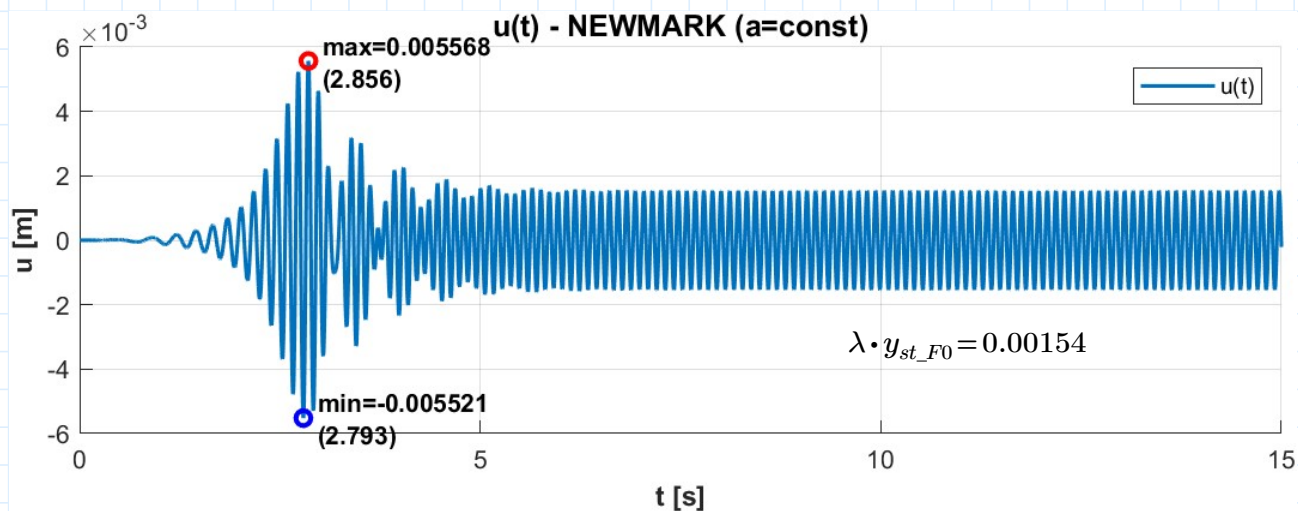
$\overline{R} := \text{APPENDPRN}("R.txt", b)$

$\max(a) = 3.71$

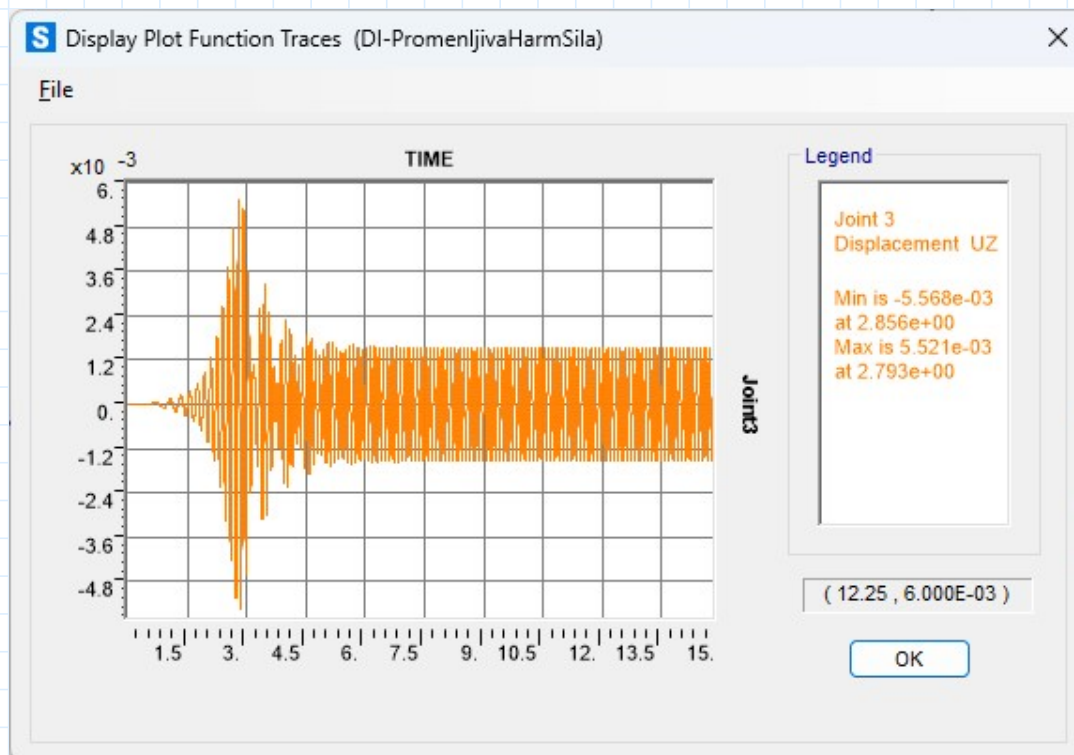
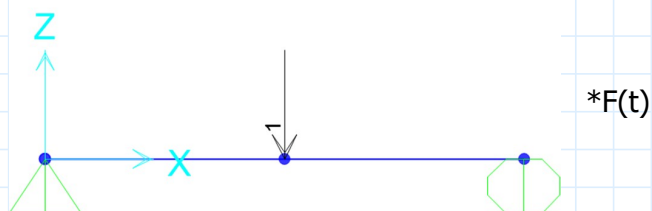
$\max(b) = 3.92$

IPE270

Numerička integracija (Matlab) Pomeranje = transient + steady state (samo od harmonijske sile)

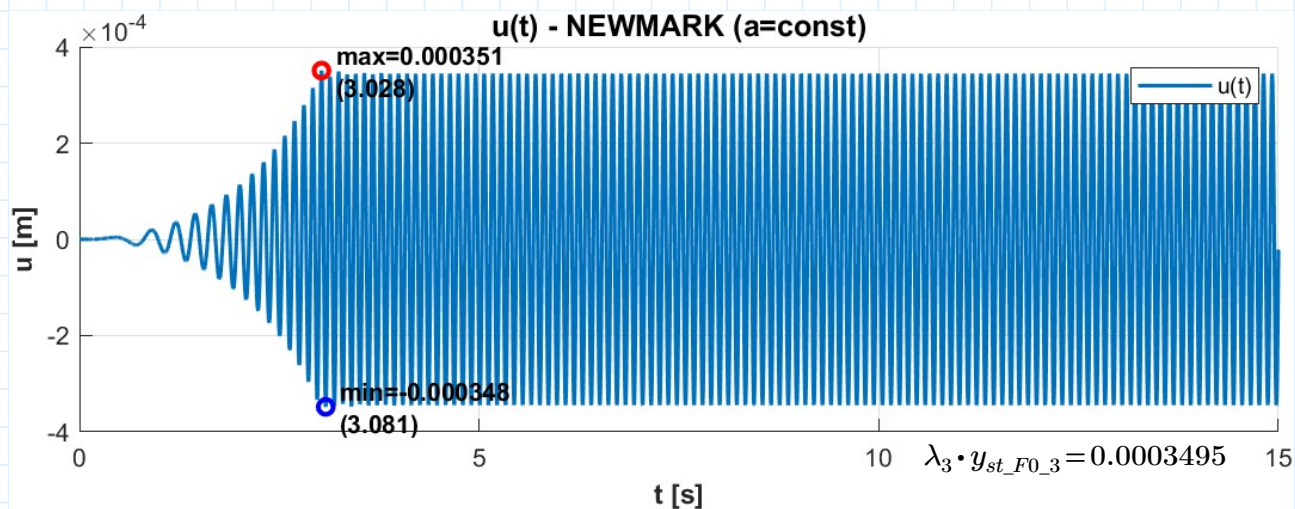


SAP2000 Pomeranje = transient + steady state (samo od harmonijske sile)



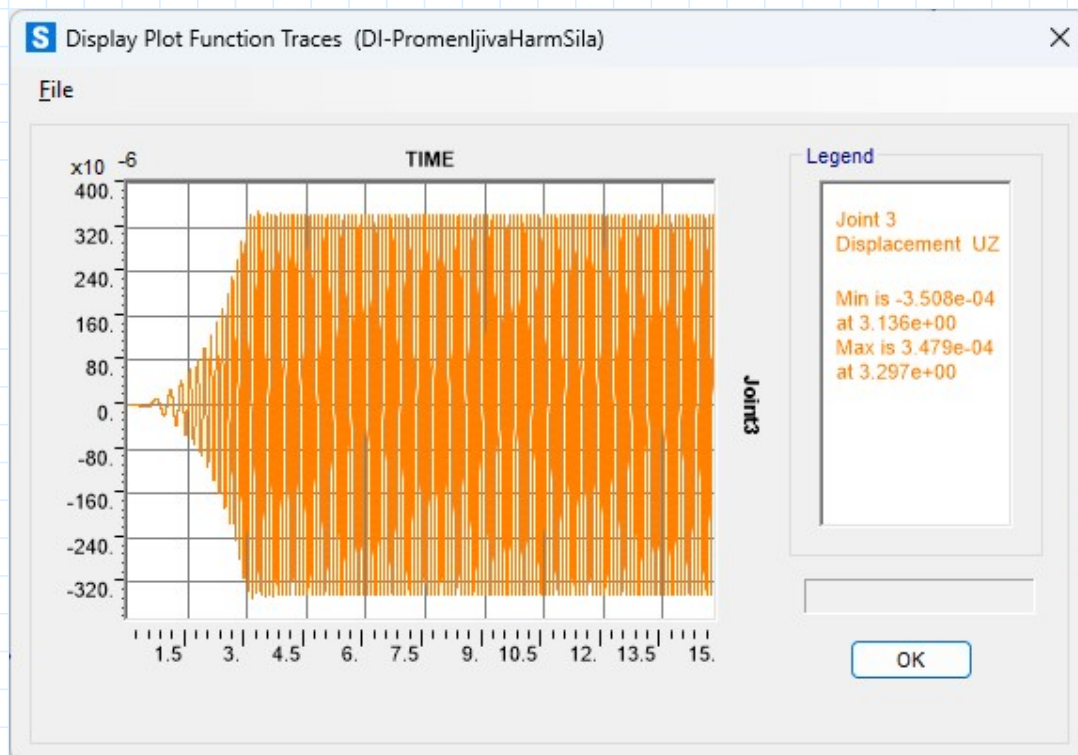
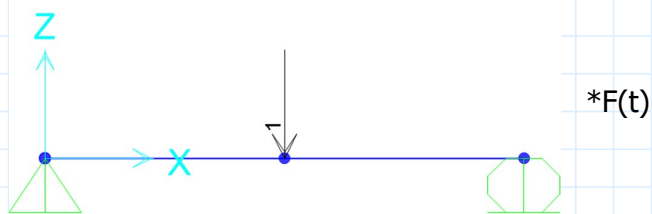
IPE400

Numerička integracija (Matlab) Pomeranje = transient + steady state (samo od harmonijske sile)

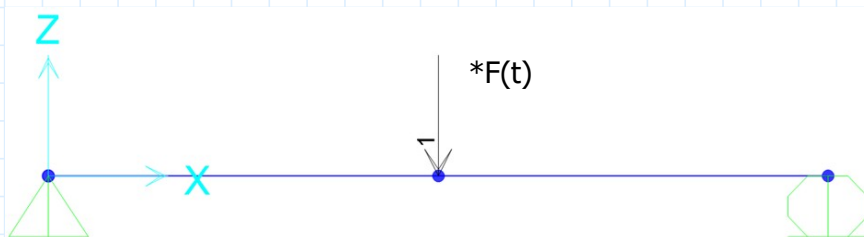


SAP2000

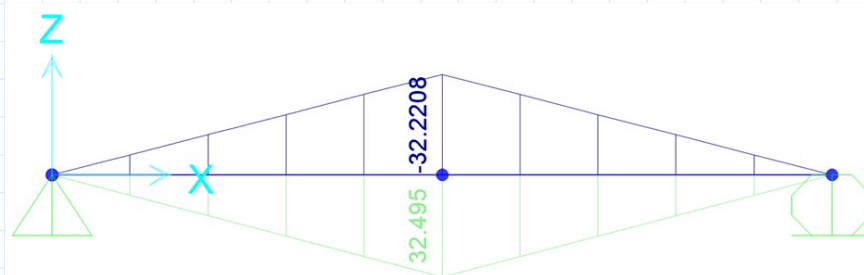
Pomeranje = transient + steady state (samo od harmonijske sile)



SAP2000 Dijagram M = transient + steady state (samo od harmonijske sile)

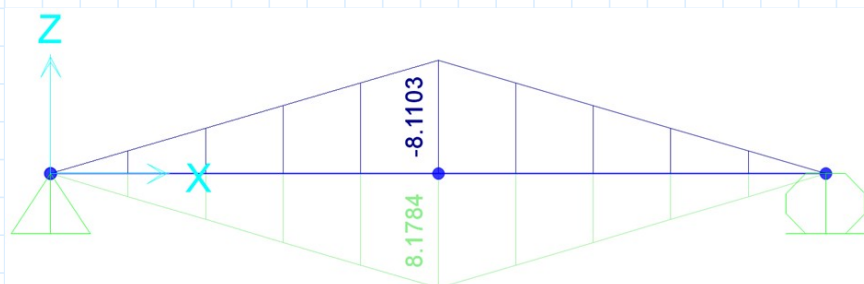


IPE 270



$$\lambda \cdot M_{st_F0} = 9.01$$

IPE 400

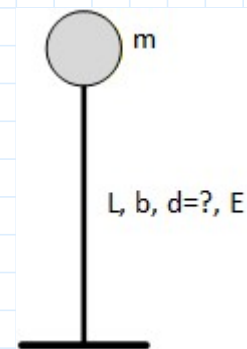


$$\lambda_3 \cdot M_{st_F0} = 8.15$$

Primer 5. Odgovor sistema usled zadatog akcelrograma

Za zadati sistem odrediti visinu pravougaonog poprečnog preseka iz uslova da je $T = 1.0$ s. Zatim odrediti odgovor sistema usled zadatog akcelrograma (txt datoteka priložena uz primer) koji deluje u horizontalnom pravcu. Homogeni početni uslovi. Analizirati samo mogućnost kretanja mase u horizontalnom pravcu. Karakteristike akcelrograma: interval zapisa 0.001 s i trajanje 20.0 s.

Podaci: $L := 2.0$ $b := 0.3$ $Em := 30 \cdot 10^6$ $m := 1.0$ $\zeta := 0.05$



Određivanje visine poprečnog preseka:

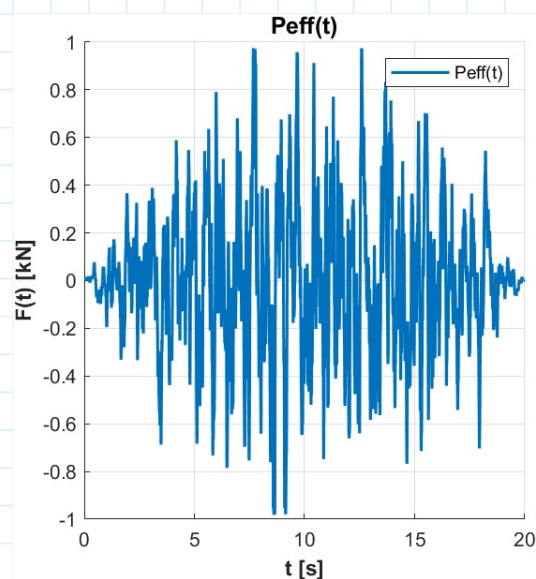
$$d_r := \sqrt[3]{\frac{3 \cdot Em \cdot \frac{b \cdot d_{rac}^3}{12}}{L^3 \cdot m}} = 1.0 \xrightarrow{\text{solve}, d_{rac}, \text{float}, 10} \begin{bmatrix} -0.0259851806 + 0.04500765304i \\ 0.0519703612 \\ -0.0259851806 - 0.04500765304i \end{bmatrix}$$

$$d := d_{r_1} = 0.0519703612$$

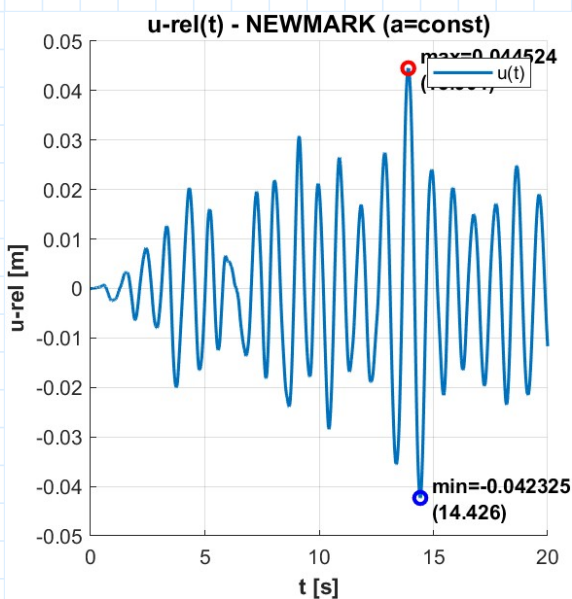
$$\text{Krutost: } k := \frac{3 \cdot Em \cdot \frac{b \cdot d^3}{12}}{L^3} = 39.4784$$

$$\text{Svojstvena kružna frekvencija: } \omega := \sqrt{\frac{k}{m}} = 6.283$$

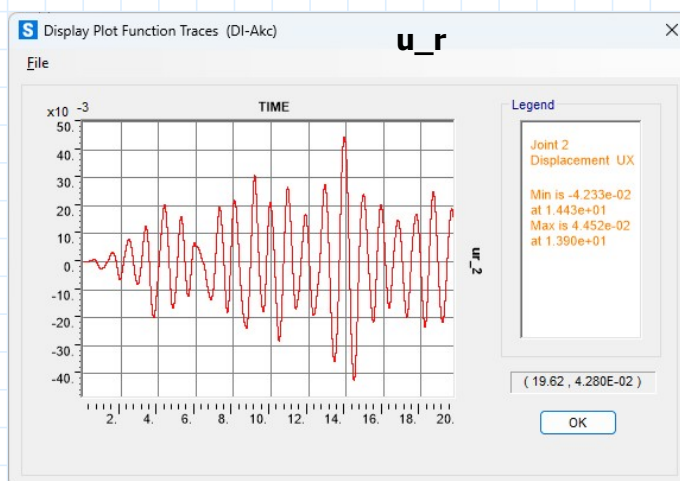
$$\text{Period svojstvenih vibracija: } T := \frac{2 \cdot \pi}{\omega} = 1.000$$

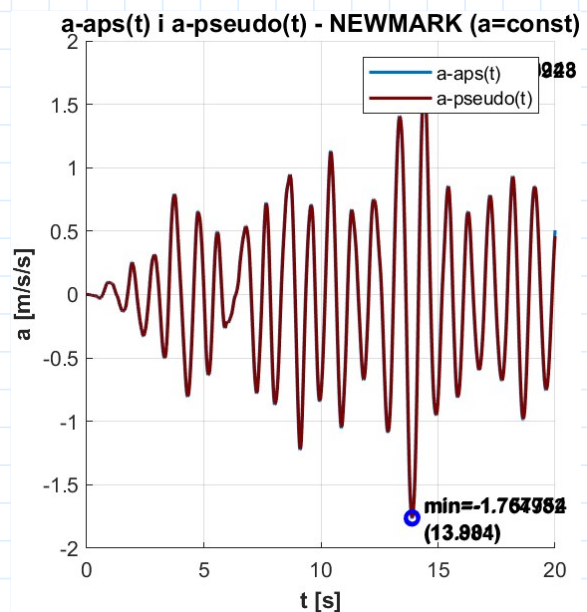
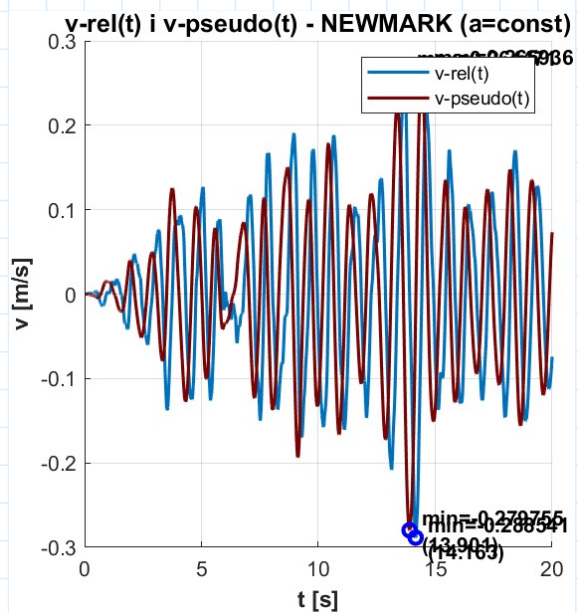
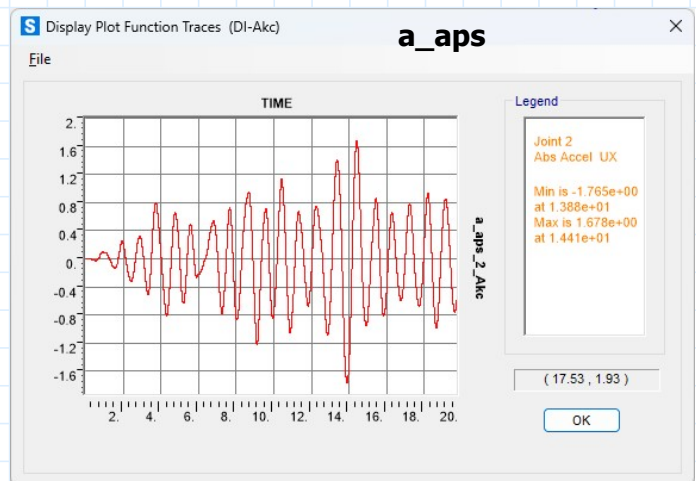
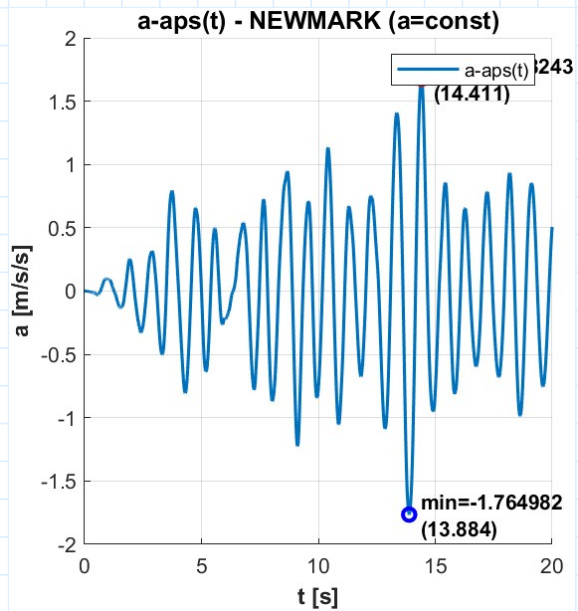
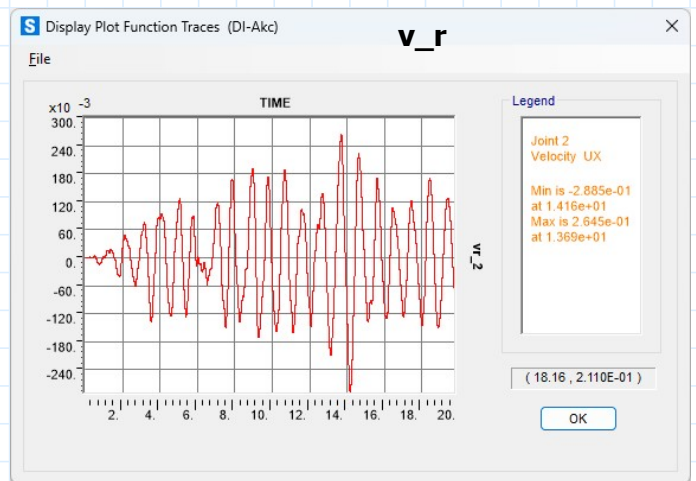
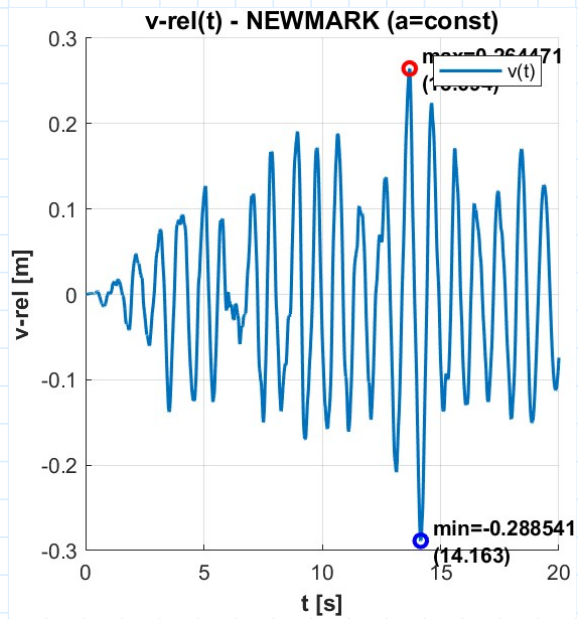


Numerička integracija (Matlab)



SAP2000





SAP2000

Dijagram M

$$u_{r_max} := 0.04452$$

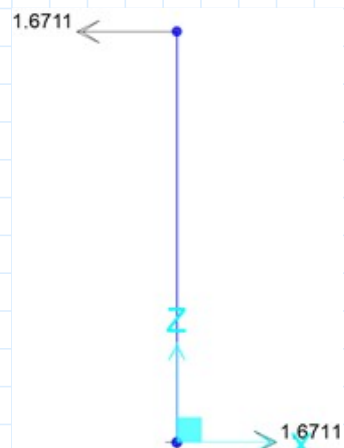
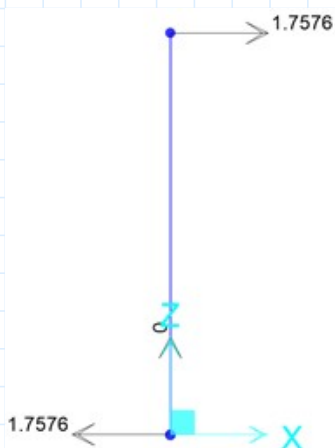
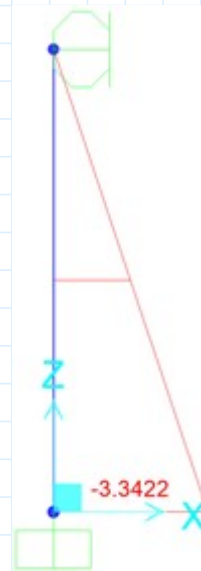
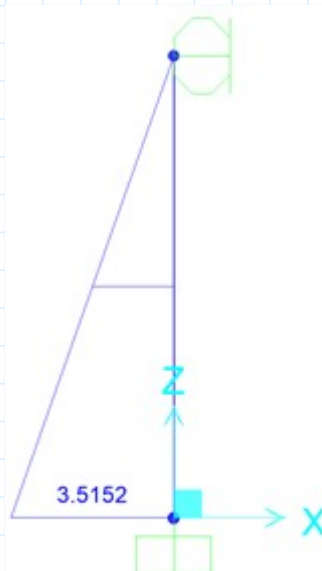
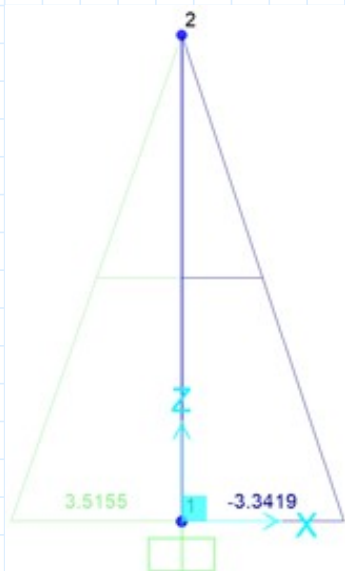
$$u_{r_min} := -0.04233$$

$$S_{max} := k \cdot u_{r_max} = 1.7576$$

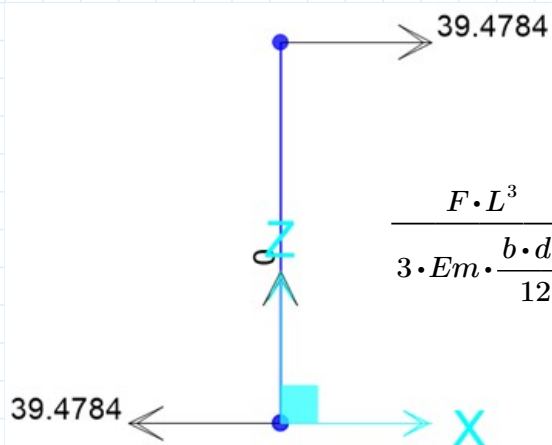
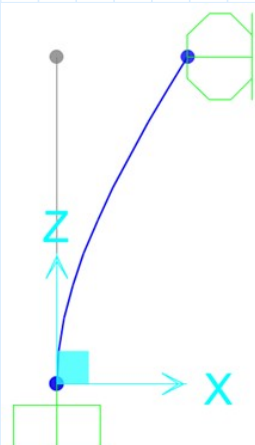
$$S_{min} := k \cdot u_{r_min} = -1.6711$$

$$M_{max} := S_{max} \cdot L = 3.5152$$

$$M_{min} := S_{min} \cdot L = -3.3422$$



Krutost (zadato co=1)



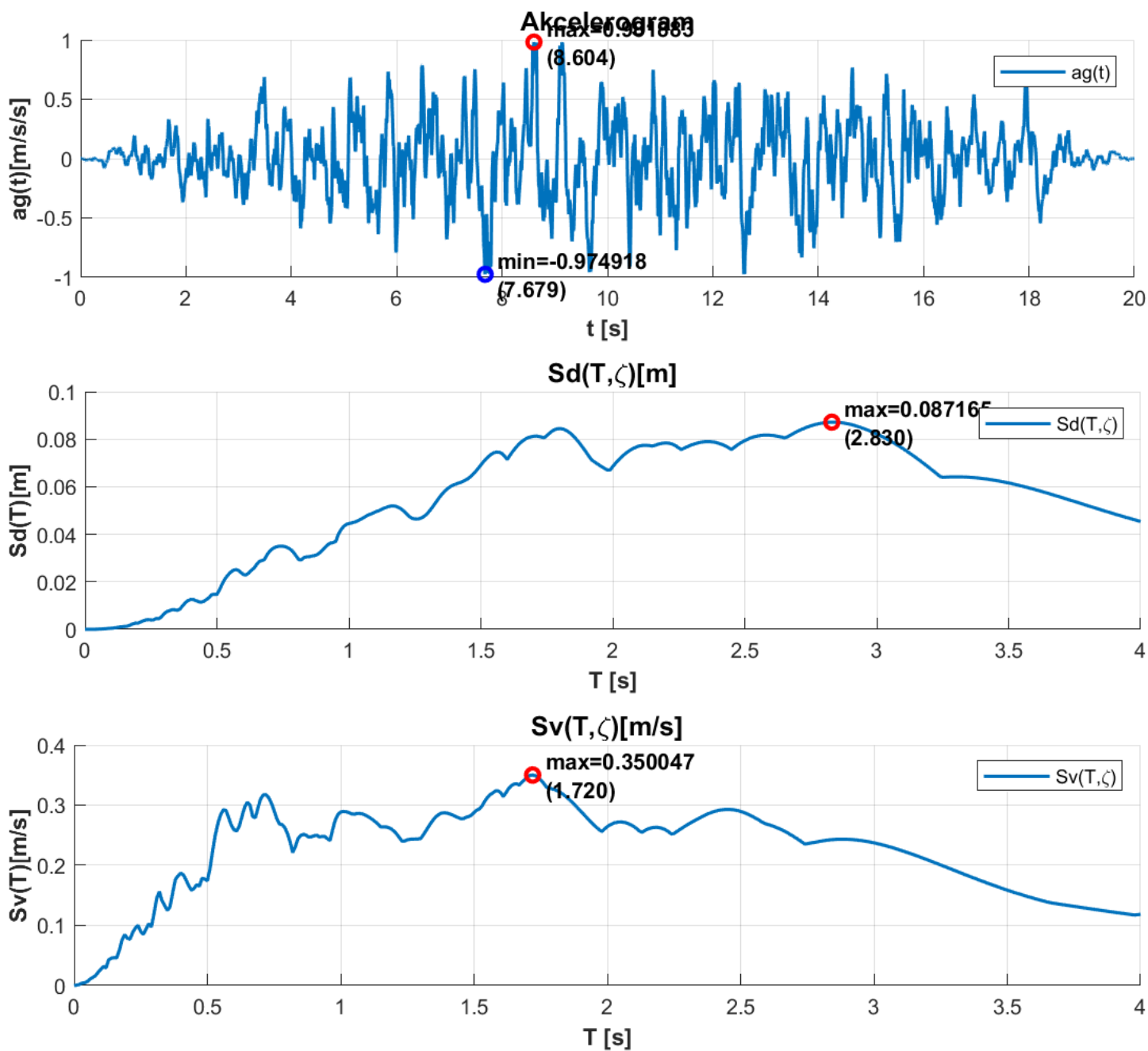
$$\frac{F \cdot L^3}{3 \cdot E m \cdot \frac{b \cdot d^3}{12}} = 1 \xrightarrow{\text{solve, F float, 6}} 39.4784$$

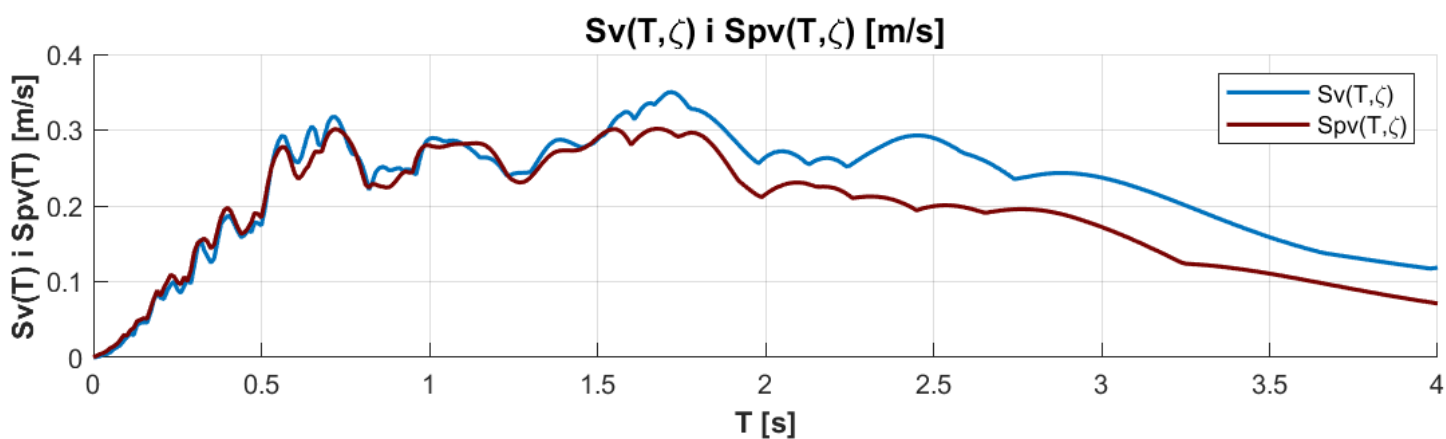
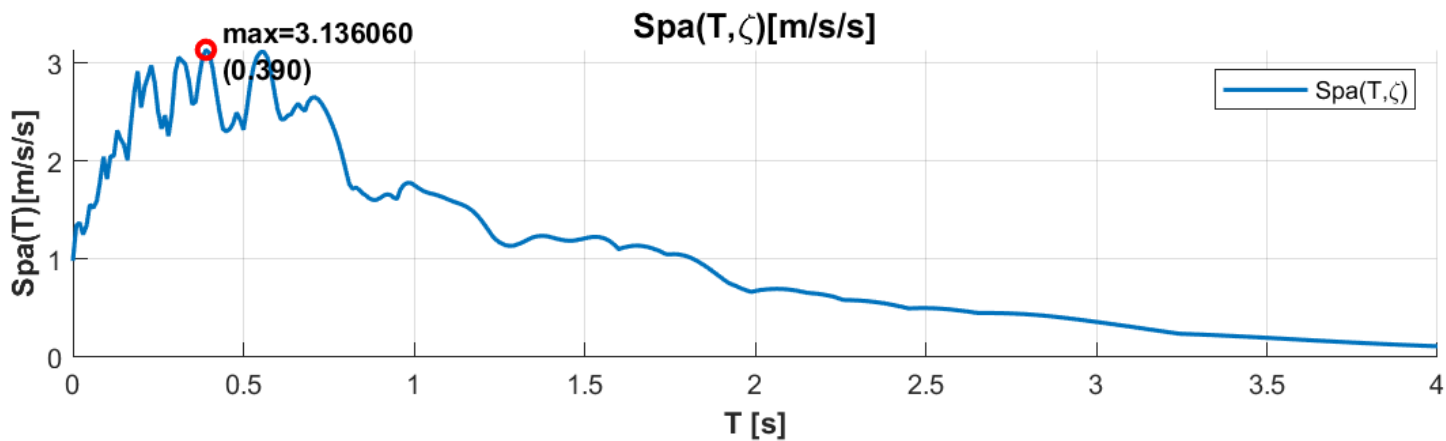
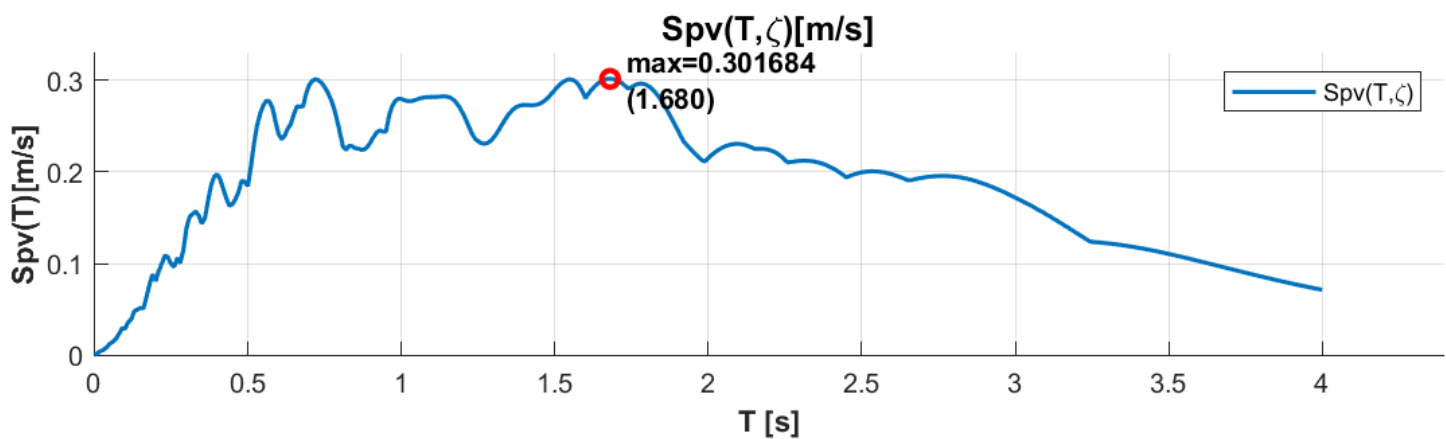
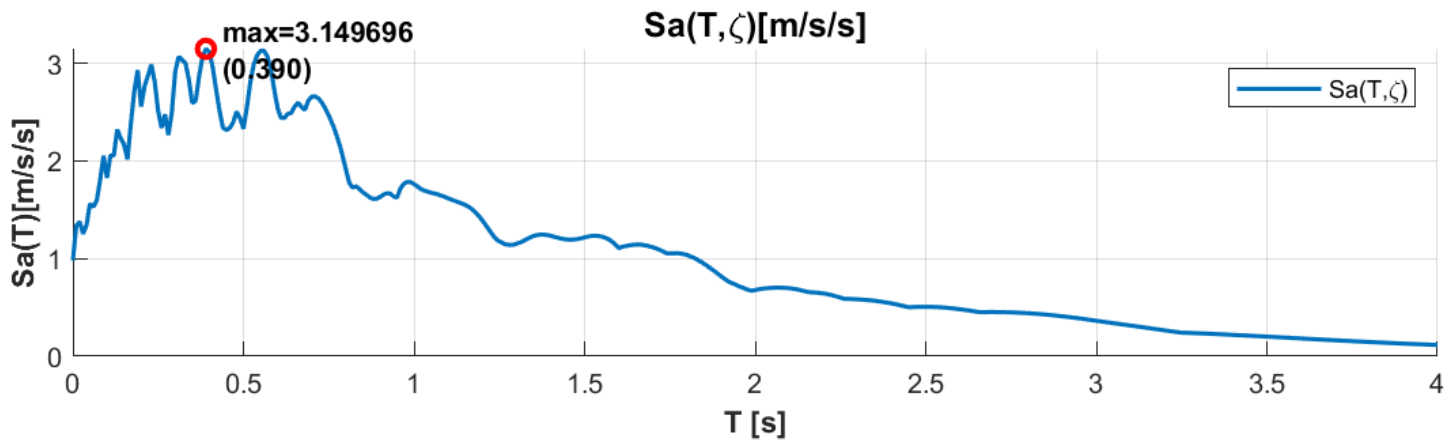
Primer 6. Spektar odgovora

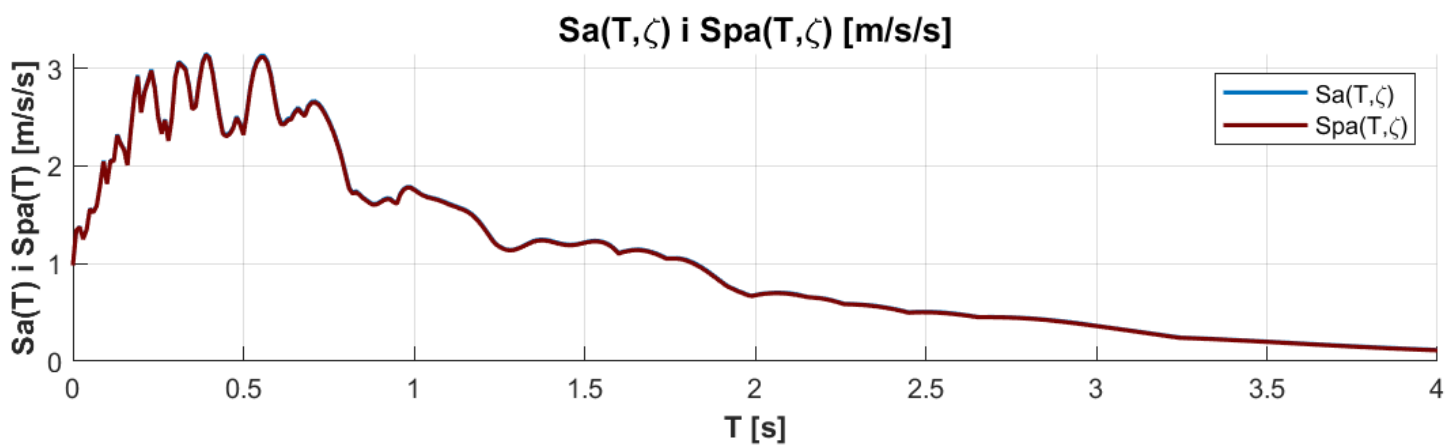
Odrediti spektre odgovora S_d , S_v , S_a , S_{pv} i S_{pa} za zadati akceleroگرام (txt datoteka priložena uz primer) i prigušenje $\zeta = 0.05$. Varirati periode svojstvenih vibracija od 0 s do 4 s (inkrement 0.1 s).

Karakteristike akcelerograma: interval zapisa 0.001 s i trajanje 20.0 s.

Numerička integracija (Matlab)







Zahtevani period: $T = 1.00000$

$Sd(1.00) = 0.04452$

$Sv(1.00) = 0.28854$

$Spv(1.00) = 0.27976$

$Sa(1.00) = 1.76498$

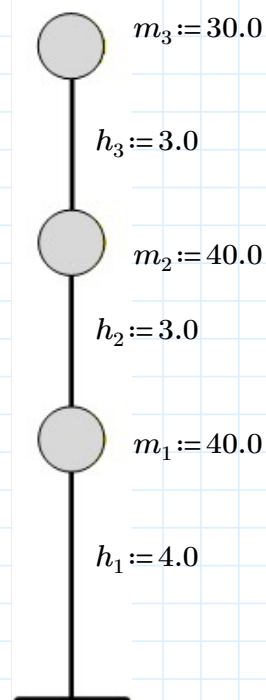
$Spa(1.00) = 1.75775$

Primer 7. MDOF. Svojstvene vrednosti

Za zadati dinamički model odrediti svojstvene vrednosti.
Mase mogu kretati samo u horizontalnom pravcu.
Podaci: $E = 33 \text{ GPa}$, $b/d = 30/100 \text{ cm}$, $m_1 = 40 \text{ t}$, $m_2 = 40 \text{ t}$, $m_3 = 30 \text{ t}$, $h_1 = 4 \text{ m}$, $h_2 = 3 \text{ m}$ i $h_3 = 3 \text{ m}$.

$$E := 33 \cdot 10^6 \quad b := 0.3 \quad d := 1.0$$

$$A := b \cdot d = 0.30 \quad I := \frac{b \cdot d^3}{12} = 0.0250$$



Statički i dinamički model

Napomena: analiza obuhvata vibracije u horizontalnom pravcu

Statički model

Stepeni slobode (pasivni):

1, 2, 3

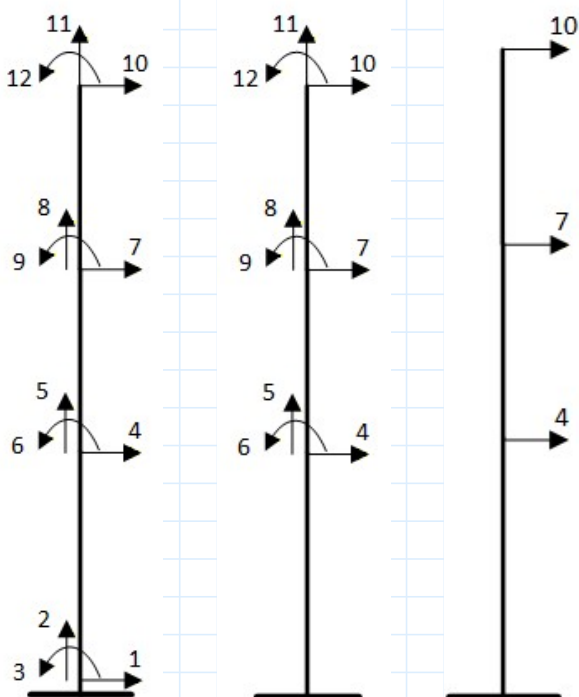
Stepeni slobode (aktivni):

4, 5, 6, 7, 8, 9, 10, 11 i 12

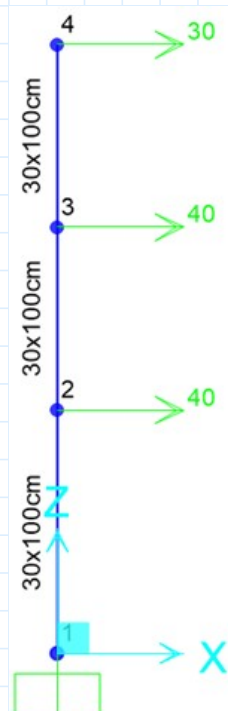
Dinamički model

Stepeni slobode (bitni):

4, 7 i 10



SAP2000



Matrica krutosti i matrica transformacije

Štap tipa k (funkcija za matricu krutosti i matricu transformacije)

$$k_k(E, L, A, I) := \frac{E \cdot I}{L^3} \cdot \begin{bmatrix} \frac{A \cdot L^2}{I} & 0 & 0 & -\frac{A \cdot L^2}{I} & 0 & 0 \\ 0 & 12 & 6 \cdot L & 0 & -12 & 6 \cdot L \\ 0 & 6 \cdot L & 4 \cdot L^2 & 0 & -6 \cdot L & 2 \cdot L^2 \\ -\frac{A \cdot L^2}{I} & 0 & 0 & \frac{A \cdot L^2}{I} & 0 & 0 \\ 0 & -12 & -6 \cdot L & 0 & 12 & -6 \cdot L \\ 0 & 6 \cdot L & 2 \cdot L^2 & 0 & -6 \cdot L & 4 \cdot L^2 \end{bmatrix}$$

$$T_k(\alpha) := \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & 0 & 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Štap 1

$$L_1 := h_1 \quad T_1 := T_k(90 \text{ deg}) \quad k_{1_LKS} := k_k(E, L_1, A, I) \quad k_{1_GKS} := T_1^T \cdot k_{1_LKS} \cdot T_1$$

$$k_{1_GKS} = \begin{bmatrix} 154687.5 & 0.0 & -309375.0 & -154687.5 & 0.0 & -309375.0 \\ 0.0 & 2475000.0 & 0.0 & 0.0 & -2475000.0 & 0.0 \\ -309375.0 & 0.0 & 825000.0 & 309375.0 & 0.0 & 412500.0 \\ -154687.5 & 0.0 & 309375.0 & 154687.5 & 0.0 & 309375.0 \\ 0.0 & -2475000.0 & 0.0 & 0.0 & 2475000.0 & 0.0 \\ -309375.0 & 0.0 & 412500.0 & 309375.0 & 0.0 & 825000.0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

Vektor stepeni slobode štapa: $ss_1 := [1 \ 2 \ 3 \ 4 \ 5 \ 6]^T$

Štap 2

$$L_2 := h_2 \quad T_2 := T_k(90 \text{ deg}) \quad k_{2_LKS} := k_k(E, L_2, A, I) \quad k_{2_GKS} := T_2^T \cdot k_{2_LKS} \cdot T_2$$

$$k_{2_GKS} = \begin{bmatrix} 366666.7 & 0.0 & -550000.0 & -366666.7 & 0.0 & -550000.0 \\ 0.0 & 3300000.0 & 0.0 & 0.0 & -3300000.0 & 0.0 \\ -550000.0 & 0.0 & 1100000.0 & 550000.0 & 0.0 & 550000.0 \\ -366666.7 & 0.0 & 550000.0 & 366666.7 & 0.0 & 550000.0 \\ 0.0 & -3300000.0 & 0.0 & 0.0 & 3300000.0 & 0.0 \\ -550000.0 & 0.0 & 550000.0 & 550000.0 & 0.0 & 1100000.0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$$

Vektor stepeni slobode štapa: $ss_2 := [4 \ 5 \ 6 \ 7 \ 8 \ 9]^T$

Štap 3

$$L_3 := h_3 \quad T_3 := T_k(90 \text{ deg}) \quad k_{3_LKS} := k_k(E, L_3, A, I) \quad k_{3_GKS} := T_3^T \cdot k_{3_LKS} \cdot T_3$$

$$k_{3_LKS} = \begin{bmatrix} & 7 & 8 & 9 & 10 & 11 & 12 \\ 3300000.0 & 0.0 & 0.0 & -3300000.0 & 0.0 & 0.0 \\ 0.0 & 366666.7 & 550000.0 & 0.0 & -366666.7 & 550000.0 \\ 0.0 & 550000.0 & 1100000.0 & 0.0 & -550000.0 & 550000.0 \\ -3300000.0 & 0.0 & 0.0 & 3300000.0 & 0.0 & 0.0 \\ 0.0 & -366666.7 & -550000.0 & 0.0 & 366666.7 & -550000.0 \\ 0.0 & 550000.0 & 550000.0 & 0.0 & -550000.0 & 1100000.0 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{bmatrix}$$

Vektor stepeni slobode štapa: $ss_3 := [7 \ 8 \ 9 \ 10 \ 11 \ 12]^T$

Kondenzovana matrica krutosti

Pasivni stepeni slobode: 1, 2 i 3

Aktivni stepeni slobode: 4, 5, 6, 7, 8, 9, 10, 11 i 12

Bitni stepeni slobode: 4, 7 i 10

$$b_s := [4 \ 7 \ 10]^T$$

Nebitni stepeni slobode: 5, 6, 8, 9, 11 i 12

$$n_s := [5 \ 6 \ 8 \ 9 \ 11 \ 12]^T$$

$$niz_{ss} := \begin{bmatrix} ss_1 \\ ss_2 \\ ss_3 \end{bmatrix} \quad niz_{mk} := \begin{bmatrix} k_{1_GKS} \\ k_{2_GKS} \\ k_{3_GKS} \end{bmatrix}$$

Funkcija za formiranje matrice sistema povezanih štapova (koristi brojeve stepeni slobode štapova i matrice krutosti štapova)

$$K(niz_{ss}, niz_{mk}, ssv, ssk) := \begin{array}{l} K_{\text{last}(ssv), \text{last}(ssk)} \leftarrow 0 \\ \text{for } m \in 0 \dots \text{last}(niz_{ss}) \\ \quad SSstap \leftarrow niz_{ss_m} \\ \quad \text{for } i \in 0 \dots \text{last}(SSstap) \\ \quad \quad \text{for } j \in 0 \dots \text{last}(ssv) \\ \quad \quad \quad \text{if } SSstap_i = ssv_j \\ \quad \quad \quad \quad \text{for } k \in 0 \dots \text{last}(SSstap) \\ \quad \quad \quad \quad \quad \text{for } l \in 0 \dots \text{last}(ssk) \\ \quad \quad \quad \quad \quad \quad \text{if } SSstap_k = ssk_l \\ \quad \quad \quad \quad \quad \quad \quad MK \leftarrow niz_{mk_m} \\ \quad \quad \quad \quad \quad \quad \quad K_{j,l} \leftarrow K_{j,l} + MK_{i,k} \\ \quad \quad \quad \quad \quad \quad \quad \text{return } K \end{array}$$

$$Kbb := K(niz_{ss}, niz_{mk}, b_s, b_s)$$

$$Kbn := K(niz_{ss}, niz_{mk}, b_s, n_s)$$

$$Knb := K(niz_{ss}, niz_{mk}, n_s, b_s)$$

$$Knn := K(niz_{ss}, niz_{mk}, n_s, n_s)$$

$$Kbb = \begin{bmatrix} 4 & 7 & 10 \\ 521354.2 & -366666.7 & 0.0 \\ -366666.7 & 733333.3 & -366666.7 \\ 0.0 & -366666.7 & 366666.7 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}$$

$$Kbn = \begin{bmatrix} 5 & 6 & 8 & 9 & 11 & 12 \\ 0.0 & -240625.0 & 0.0 & -550000.0 & 0.0 & 0.0 \\ 0.0 & 550000.0 & 0.0 & 0.0 & 0.0 & -550000.0 \\ 0.0 & 0.0 & 0.0 & 550000.0 & 0.0 & 550000.0 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}$$

$$Knb = \begin{bmatrix} 4 & 7 & 10 \\ 0.0 & 0.0 & 0.0 \\ -240625.0 & 550000.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ -550000.0 & 0.0 & 550000.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & -550000.0 & 550000.0 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 8 \\ 9 \\ 11 \\ 12 \end{bmatrix}$$

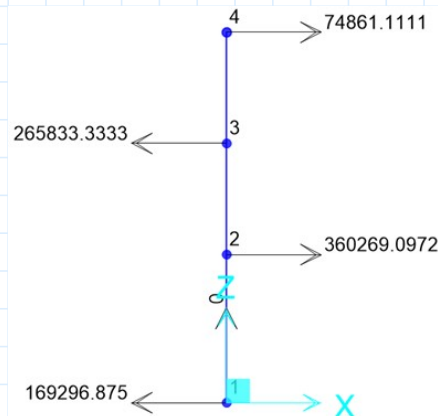
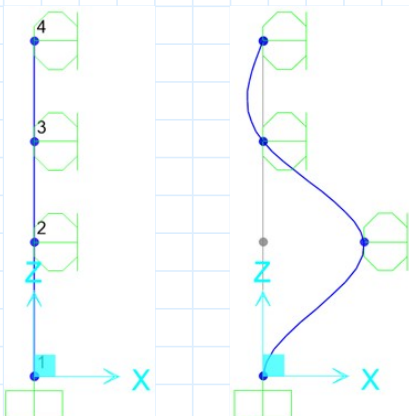
$$\text{if } (Knb = Kbn^T, \text{"OK"}, \text{"NO OK"}) = \text{"OK"}$$

$$Knn = \begin{bmatrix} 5 & 6 & 8 & 9 & 11 & 12 \\ 5775000.0 & 0.0 & -3300000.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1925000.0 & 0.0 & 550000.0 & 0.0 & 0.0 \\ -3300000.0 & 0.0 & 6600000.0 & 0.0 & -3300000.0 & 0.0 \\ 0.0 & 550000.0 & 0.0 & 2200000.0 & 0.0 & 550000.0 \\ 0.0 & 0.0 & -3300000.0 & 0.0 & 3300000.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 550000.0 & 0.0 & 1100000.0 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 8 \\ 9 \\ 11 \\ 12 \end{bmatrix}$$

$$Kc := Kbb - Kbn \cdot Knn^{-1} \cdot Knb = \begin{bmatrix} 4 & 7 & 10 \\ 360269.097 & -265833.333 & 74861.111 \\ -265833.333 & 293333.333 & -110000.000 \\ 74861.111 & -110000.000 & 48888.889 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}$$

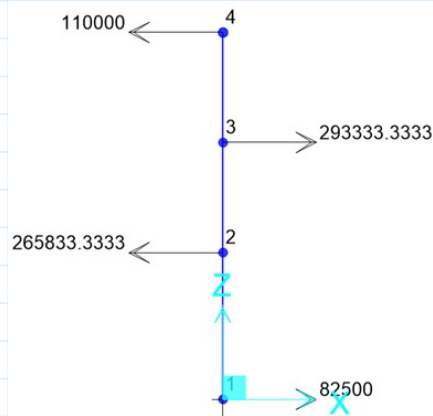
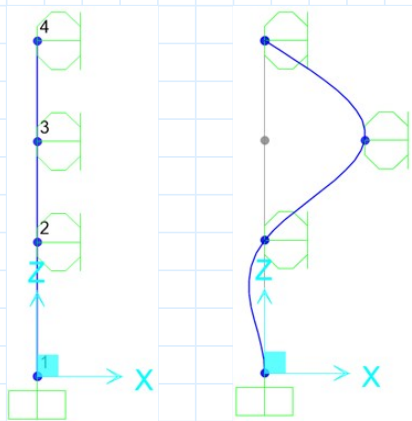
SAP2000

Zadato c2=1



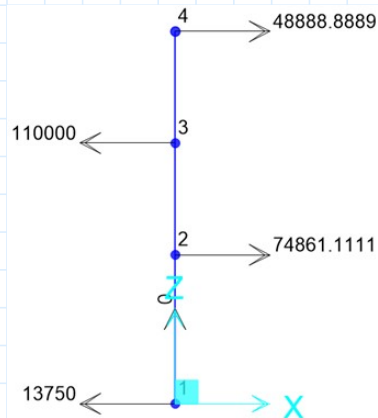
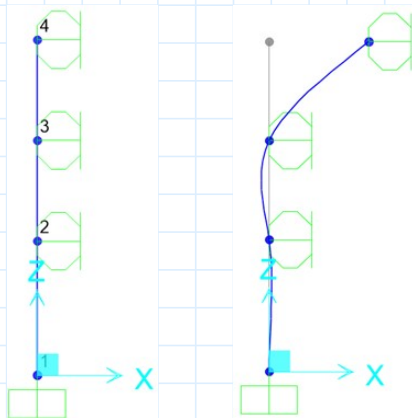
$$Kc^{(0)} = \begin{bmatrix} 360269.097 \\ -265833.333 \\ 74861.111 \end{bmatrix}$$

Zadato c3=1



$$Kc^{(1)} = \begin{bmatrix} -265833.333 \\ 293333.333 \\ -110000.000 \end{bmatrix}$$

Zadato c4=1



$$Kc^{(2)} = \begin{bmatrix} 74861.111 \\ -110000.000 \\ 48888.889 \end{bmatrix}$$

Matrica mase

$$Mc := \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 10 \\ 40.0 & 0.0 & 0.0 \\ 0.0 & 40.0 & 0.0 \\ 0.0 & 0.0 & 30.0 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}$$

Kružne frekvencije i periodi svojstvenih vibracija

$$\lambda_{rac} := \text{eigenvals}(Mc^{-1} \cdot Kc) = \begin{bmatrix} 15846.65055 \\ 2067.95045 \\ 55.08939 \end{bmatrix}$$

$$\omega_{rac} := \sqrt{\lambda_{rac}} = \begin{bmatrix} 125.88348 \\ 45.47472 \\ 7.42222 \end{bmatrix}$$

$$T_{rac} := \frac{2 \cdot \pi}{\omega_{rac}} = \begin{bmatrix} 0.04991 \\ 0.13817 \\ 0.84654 \end{bmatrix}$$

$$T_{ton_1} := T_{rac_2} = 0.84654$$

$$T_{ton_2} := T_{rac_1} = 0.13817$$

$$T_{ton_3} := T_{rac_0} = 0.04991$$

Komentar:

$$\lambda_{rac_2} := \text{eigenvals}(Kc^{-1} \cdot Mc) = \begin{bmatrix} 0.01815231 \\ 0.00048357 \\ 0.00006310 \end{bmatrix}$$

$$\omega_{rac_2} := \sqrt{\frac{1}{\lambda_{rac_2}}} = \begin{bmatrix} 7.42222 \\ 45.47472 \\ 125.88348 \end{bmatrix}$$

$$T_{rac_2} := \frac{2 \cdot \pi}{\omega_{rac_2}} = \begin{bmatrix} 0.84654 \\ 0.13817 \\ 0.04991 \end{bmatrix}$$

Frekventna jednačina - $\det(K - \omega^2 M)=0$; $\det(M^{-1} \cdot K - \omega^2 I)=0$; $(KA - \omega^2 MA=0)$

$$I_{mat} := \text{identity}(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Karakteristična determinanta} \\ kPol_1(\lambda_1) := \det(Mc^{-1} \cdot Kc - \lambda_1 \cdot I_{mat})$$

Karakteristični polinom

$$kPol_1(\lambda_1) \xrightarrow[\text{float, 6}]{\text{simplify}} -1.0 \cdot \lambda_1^3 + 17969.7 \cdot \lambda_1^2 - 0.33757 \cdot 10^8 \cdot \lambda_1 + 0.180528 \cdot 10^{10}$$

$$kcoef_1 := kPol_1(\lambda_1) \xrightarrow[\text{float, 6}]{\text{coeffs}} \begin{bmatrix} 0.180528 \cdot 10^{10} \\ -(0.33757 \cdot 10^8) \\ 17969.7 \\ -1.0 \end{bmatrix} \quad \text{Koreni karakterističnog polinoma } (\omega^2) \\ skf_1 := \sqrt{\text{polyroots}(kcoef_1)} = \begin{bmatrix} 7.42221 \\ 45.47471 \\ 125.88352 \end{bmatrix}$$

Frekventna jednačina - $\det(DM - \frac{1}{\omega^2} I)=0$; $\det(K^{-1} \cdot M - \frac{1}{\omega^2} I)=0$; $(A - DM \omega^2 A=0)$

$$kPol_2(\lambda) := \det(Kc^{-1} \cdot Mc - \lambda \cdot I_{mat})$$

$$kPol_2(\lambda) \xrightarrow[\text{float, 6}]{\text{simplify}} -1.0 \cdot \lambda^3 + 0.018699 \cdot \lambda^2 - 0.00000995394 \cdot \lambda + 0.553929 \cdot 10^{-9}$$

$$kcoef_2 := kPol_2(\lambda) \xrightarrow[\text{float, 6}]{\text{coeffs}} \begin{bmatrix} 0.553929 \cdot 10^{-9} \\ -0.00000995394 \\ 0.018699 \\ -1.0 \end{bmatrix} \quad \text{Koreni karakterističnog polinoma } (\frac{1}{\omega^2}) \\ \frac{1}{\sqrt{\text{polyroots}(kcoef_2)}} = \begin{bmatrix} 125.88352917 \\ 45.47473393 \\ 7.42222084 \end{bmatrix}$$

Oblici svojstvenih vibracija

$$A_{rac} := \text{eigenvecs}(Mc^{-1} \cdot Kc) = \begin{bmatrix} -0.70651 & 0.63972 & 0.1884 \\ 0.64535 & 0.50467 & 0.49297 \\ -0.29045 & -0.57972 & 0.8494 \end{bmatrix}$$

$$\text{Ton 1: } A_{rac_1} := A_{rac}^{(2)} = \begin{bmatrix} 0.18840 \\ 0.49297 \\ 0.84940 \end{bmatrix} \quad \text{Ton 2: } A_{rac_2} := A_{rac}^{(1)} = \begin{bmatrix} 0.63972 \\ 0.50467 \\ -0.57972 \end{bmatrix} \quad \text{Ton 3: } A_{rac_3} := -A_{rac}^{(0)} = \begin{bmatrix} 0.70651 \\ -0.64535 \\ 0.29045 \end{bmatrix}$$

Komentar:

$$\text{eigenvecs}(Kc^{-1} \cdot Mc) = \begin{bmatrix} -0.1884 & -0.63972 & -0.70651 \\ -0.49297 & -0.50467 & 0.64535 \\ -0.8494 & 0.57972 & -0.29045 \end{bmatrix}$$

Ortonormiranje svojstvenih oblika

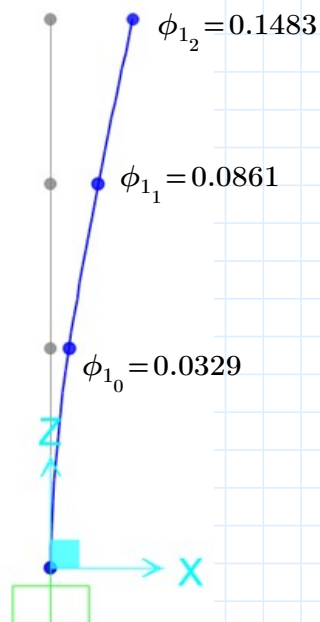
Ton 1 $M_{gen_1} := A_{rac_1}^T \cdot Mc \cdot A_{rac_1} = 32.78514$ $\phi_1 := \frac{1}{\sqrt{M_{gen_1}}} \cdot A_{rac_1} = \begin{bmatrix} 0.03290 \\ 0.08610 \\ 0.14835 \end{bmatrix}$
 $M_1 := \phi_1^T \cdot Mc \cdot \phi_1 = 1.00000$

Ton 2 $M_{gen_2} := A_{rac_2}^T \cdot Mc \cdot A_{rac_2} = 36.63930$ $\phi_2 := \frac{1}{\sqrt{M_{gen_2}}} \cdot A_{rac_2} = \begin{bmatrix} 0.10569 \\ 0.08337 \\ -0.09577 \end{bmatrix}$
 $M_2 := \phi_2^T \cdot Mc \cdot \phi_2 = 1.00000$

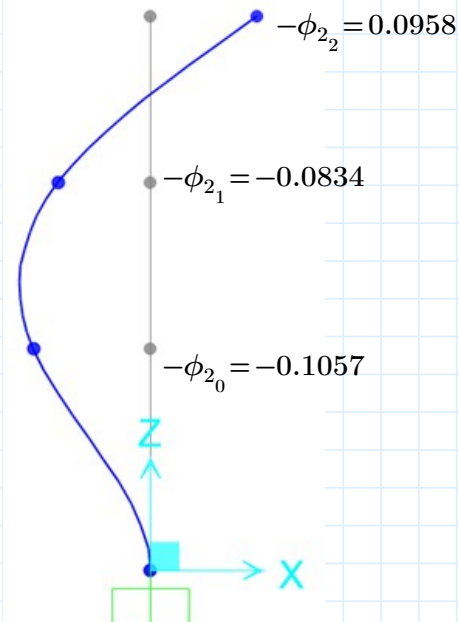
Ton 3 $M_{gen_3} := A_{rac_3}^T \cdot Mc \cdot A_{rac_3} = 39.15640$ $\phi_3 := \frac{1}{\sqrt{M_{gen_3}}} \cdot A_{rac_3} = \begin{bmatrix} 0.11291 \\ -0.10313 \\ 0.04642 \end{bmatrix}$
 $M_3 := \phi_3^T \cdot Mc \cdot \phi_3 = 1.00000$

SAP2000

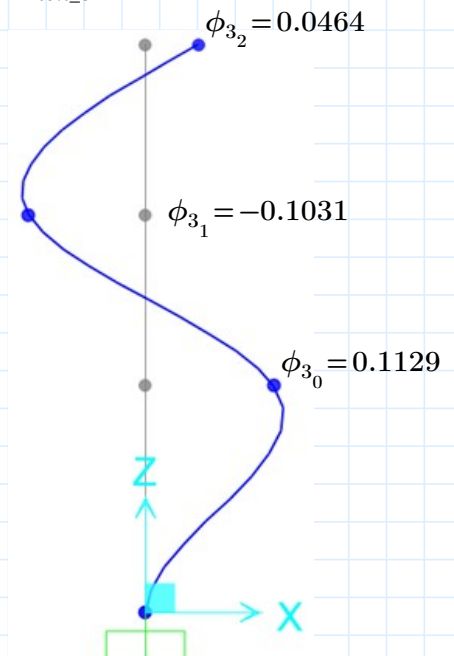
$T_{ton_1} = 0.847$



$T_{ton_2} = 0.138$



$T_{ton_3} = 0.050$



OutputCase	StepType Text	StepNum Unitless	Period Sec	Frequency Cyc/sec	CircFreq rad/sec
MODAL	Mode	1	0.846537	1.18128347...	7.42222296...
MODAL	Mode	2	0.138169	7.23752698...	45.4747232...
MODAL	Mode	3	0.049913	20.0349781...	125.883480...

Mode	1	0.032904
Mode	2	-0.105686
Mode	3	0.112906
Mode	1	0.086096
Mode	2	-0.083374
Mode	3	-0.103133
Mode	1	0.148346
Mode	2	0.095773
Mode	3	0.046416

Oblik vibracija prvog tona ($K \cdot A_1 = \omega_1^2 M \cdot A_1$)

$$\text{sistem}(A_{2_1}, A_{3_1}) := (Kc - skf_{1_0}^2 \cdot Mc) \cdot \begin{bmatrix} 1.0 \\ A_{2_1} \\ A_{3_1} \end{bmatrix}$$

Komentar:
Usvoji se jedna vrednost u vektoru

$$\text{sistem}(A_{2_1}, A_{3_1})_{1,0} \xrightarrow[\text{float, 10}]{\text{expand}} -110000.0 \cdot A_{3_1} + 291129.7635 \cdot A_{2_1} - 265833.3333$$

$$\text{sistem}(A_{2_1}, A_{3_1})_{2,0} \xrightarrow[\text{float, 10}]{\text{expand}} 47236.21148 \cdot A_{3_1} - 110000.0 \cdot A_{2_1} + 74861.11111$$

Solver	Guess Values	$A_{2_1} := 1 \quad A_{3_1} := 1$
	Constraints	$\text{sistem}(A_{2_1}, A_{3_1})_{1,0} = 0 \quad \text{sistem}(A_{2_1}, A_{3_1})_{2,0} = 0$
		$\begin{bmatrix} y \\ z \end{bmatrix} := \text{find}(A_{2_1}, A_{3_1}) = \begin{bmatrix} 2.61658 \\ 4.50846 \end{bmatrix}$

$$\begin{bmatrix} 1.0 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1.00000 \\ 2.61658 \\ 4.50846 \end{bmatrix}$$

Prethodno rešenje:

$$\frac{1}{\phi_{1_0}} \cdot \phi_1 = \begin{bmatrix} 1.00000 \\ 2.61658 \\ 4.50846 \end{bmatrix}$$

Komentar:

Analognim postupkom se određuju oblici vibracija i za ostale tonove

Svojstvene vrednosti. Matlab

```
Kbb =
    521354.167   -366666.667         0.000
   -366666.667    733333.333   -366666.667
         0.000   -366666.667    366666.667

Kbn =
    0.000  -240625.000    -0.000  -550000.000         0.000         0.000
   -0.000   550000.000         0.000         0.000   -0.000  -550000.000
    0.000         0.000    -0.000   550000.000         0.000   550000.000

Knbb =
    0.000    -0.000         0.000
  -240625.000   550000.000         0.000
    -0.000         0.000    -0.000
 -550000.000         0.000   550000.000
    0.000    -0.000         0.000
    0.000  -550000.000   550000.000

Knn =
    5775000.000         0.000  -3300000.000         0.000         0.000         0.000
         0.000   1925000.000         -0.000   550000.000         0.000         0.000
  -3300000.000         -0.000   6600000.000         0.000  -3300000.000         0.000
         0.000   550000.000         0.000  2200000.000        -0.000   550000.000
         0.000         0.000  -3300000.000        -0.000  3300000.000        -0.000
         0.000         0.000         0.000   550000.000        -0.000  1100000.000

Kc =
    360269.097  -265833.333    74861.111
   -265833.333    293333.333   -110000.000
    74861.111   -110000.000    48888.889
```

```
Modalna matrica iz
[modalnaMatrica, spektralnaMatrica]=eig(Kc,Mc)
A =
0.22181    1.00000    1.00000
0.58037    0.78888   -0.91343
1.00000   -0.90620    0.41110
```

Modálna matica ortonormálna

```
Fi =
0.03290    0.10569    0.11291
0.08610    0.08337   -0.10313
0.14835   -0.09577    0.04642
```

Spektralna matrica

(kvadrati svojstvenih kruznih frekvencija) =

55.08939	0.00000	0.00000
0.00000	2067.95045	0.00000
0.00000	0.00000	15846.65055

Svojstvene kruzne frekvencije

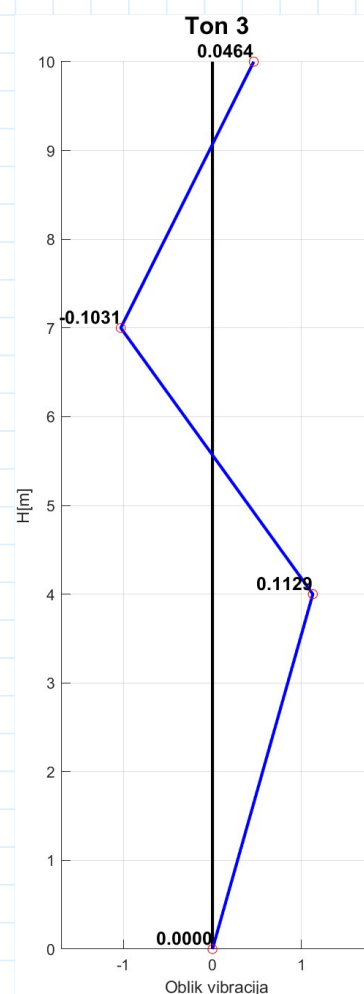
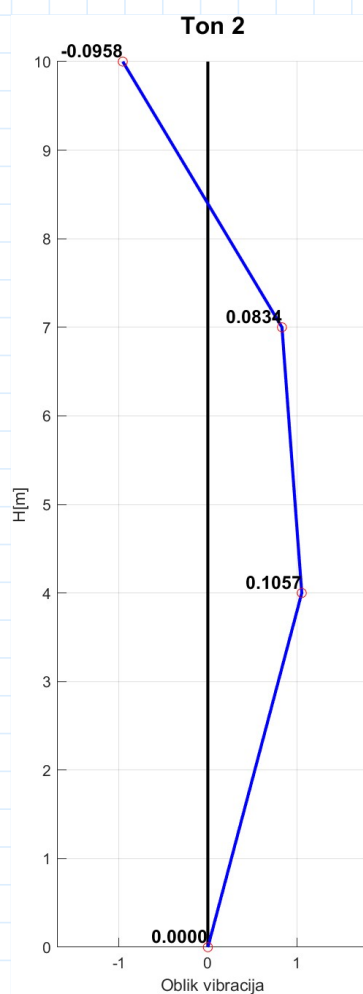
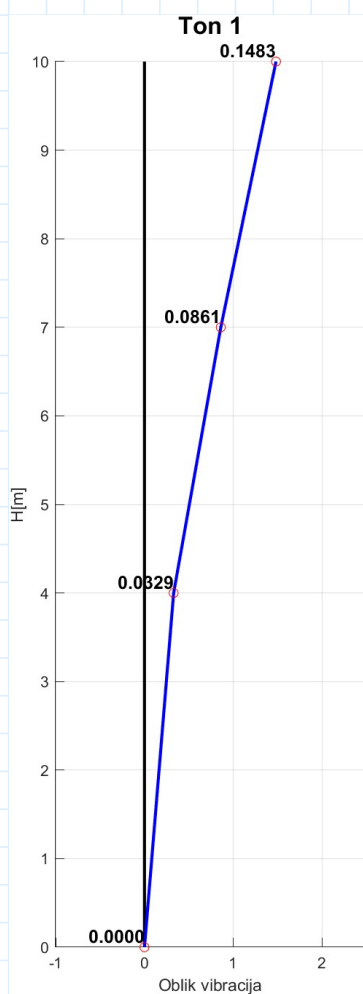
 $\omega =$

7.42222	0.00000	0.00000
0.00000	45.47472	0.00000
0.00000	0.00000	125.88348

Svojstveni periodi vibracija

$$T =$$

0.84654	0.00000	0.00000
0.00000	0.13817	0.00000
0.00000	0.00000	0.04991



```

Frekventna jednacina - det(K-w^2M)=0; det(K-lambda*M)=0
Karakteristicni polinom:
      8.62545e+8*lambda^2 - 1.62034e+12*lambda - 48000.0*lambda^3 + 8.66536e+13
Koreni karakteristicnog polinoma
      15846.65055
      2067.95045
      55.08939
-----
Frekventna jednacina - det(DM-(1/w^2)I)=0; det(K^-1*M-(1/w^2)I)=0; det(K^-1*M-lambda*I)=0
Karakteristicni polinom:
      0.0186989899*lambda^2 - 0.000009953939394*lambda - 1.0*lambda^3 + 0.0000000005539293764
Koreni karakteristicnog polinoma
      0.0181523145
      0.0004835706
      0.0000631048

```

ITERATIVNI POSTUPAK

Dinamička matrica: $D_M := Kc^{-1} \cdot Mc$ Početna pretpostavka: $B_0 := \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Iteracije za prvi ton

$$I_{1_1} := D_M \cdot B_0 = \begin{bmatrix} 0.01552 \\ 0.04061 \\ 0.07006 \end{bmatrix} \quad \lambda_{1_1} := I_{1_1_0} = 0.01551515 \quad B_{1_1} := \frac{1}{I_{1_1_0}} \cdot I_{1_1} = \begin{bmatrix} 1.00000 \\ 2.61719 \\ 4.51562 \end{bmatrix}$$

$$I_{1_2} := D_M \cdot B_{1_1} = \begin{bmatrix} 0.01817 \\ 0.04755 \\ 0.08193 \end{bmatrix} \quad \lambda_{1_2} := I_{1_2_0} = 0.01817172 \quad B_{1_2} := \frac{1}{I_{1_2_0}} \cdot I_{1_2} = \begin{bmatrix} 1.00000 \\ 2.61666 \\ 4.50873 \end{bmatrix}$$

$$I_{1_3} := D_M \cdot B_{1_2} = \begin{bmatrix} 0.01815 \\ 0.04750 \\ 0.08184 \end{bmatrix} \quad \lambda_{1_3} := I_{1_3_0} = 0.01815318 \quad B_{1_3} := \frac{1}{I_{1_3_0}} \cdot I_{1_3} = \begin{bmatrix} 1.00000 \\ 2.61658 \\ 4.50847 \end{bmatrix}$$

$$I_{1_4} := D_M \cdot B_{1_3} = \begin{bmatrix} 0.01815 \\ 0.04750 \\ 0.08184 \end{bmatrix} \quad \lambda_{1_4} := I_{1_4_0} = 0.01815234 \quad B_{1_4} := \frac{1}{I_{1_4_0}} \cdot I_{1_4} = \begin{bmatrix} 1.00000 \\ 2.61658 \\ 4.50846 \end{bmatrix}$$

$$I_{1_5} := D_M \cdot B_{1_4} = \begin{bmatrix} 0.01815 \\ 0.04750 \\ 0.08184 \end{bmatrix} \quad \lambda_{1_5} := I_{1_5_0} = 0.01815232 \quad B_{1_5} := \frac{1}{I_{1_5_0}} \cdot I_{1_5} = \begin{bmatrix} 1.00000 \\ 2.61658 \\ 4.50846 \end{bmatrix}$$

$$\frac{1}{\omega_{1_{iter}}^2} = \lambda_{1_5} \xrightarrow[\text{float, 6}]{\text{solve, } \omega_{1_{iter}}} \begin{bmatrix} -7.42222 \\ 7.42222 \end{bmatrix} \quad A_1 := B_{1_5} = \begin{bmatrix} 1.00000 \\ 2.61658 \\ 4.50846 \end{bmatrix}$$

Prethodno rešenje: $\omega_{rac_{2_0}} = 7.42222$ $\frac{1}{\phi_{1_0}} \cdot \phi_1 = \begin{bmatrix} 1.00000 \\ 2.61658 \\ 4.50846 \end{bmatrix}$

Matrica brisanja prvog tona

$$orto_1 := A_1^T \cdot Mc \cdot \begin{bmatrix} A_{1_2} \\ A_{2_2} \\ A_{3_2} \end{bmatrix} = 0 \xrightarrow[\text{float}, 10]{\text{solve}, A_{1_2}} -3.381344973 \cdot A_{3_2} - 2.616578723 \cdot A_{2_2}$$

$$S_1 := \begin{bmatrix} 0 & -2.616578723 & -3.381344973 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} A1_2 = 0 \cdot A1_2 - 2.6166 \cdot A2_2 - 3.3813 \cdot A3_2 \\ A2_2 = 0 \cdot A1_2 + 1 \cdot A2_2 + 0 \cdot A3_2 \\ A3_2 = 0 \cdot A1_2 + 0 \cdot A2_2 + 1 \cdot A3_2 \end{array}$$

Modifikovana dinamička matrica: $D_{M_1} := D_M \cdot S_1$

Iteracije za drugi ton

Početna pretpostavka: $B_{0_2} := \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$I_{2_1} := D_{M_1} \cdot B_{0_2} = \begin{bmatrix} 0.00047 \\ 0.00039 \\ -0.00044 \end{bmatrix} \quad \lambda_{2_1} := I_{2_1_0} = 0.00046780 \quad B_{2_1} := \frac{1}{I_{2_1_0}} \cdot I_{2_1} = \begin{bmatrix} 1.00000 \\ 0.84240 \\ -0.94761 \end{bmatrix}$$

$$I_{2_2} := D_{M_1} \cdot B_{2_1} = \begin{bmatrix} 0.00050 \\ 0.00040 \\ -0.00045 \end{bmatrix} \quad \lambda_{2_2} := I_{2_2_0} = 0.00049679 \quad B_{2_2} := \frac{1}{I_{2_2_0}} \cdot I_{2_2} = \begin{bmatrix} 1.00000 \\ 0.79568 \\ -0.91146 \end{bmatrix}$$

$$I_{2_3} := D_{M_1} \cdot B_{2_2} = \begin{bmatrix} 0.00049 \\ 0.00038 \\ -0.00044 \end{bmatrix} \quad \lambda_{2_3} := I_{2_3_0} = 0.00048525 \quad B_{2_3} := \frac{1}{I_{2_3_0}} \cdot I_{2_3} = \begin{bmatrix} 1.00000 \\ 0.78977 \\ -0.90689 \end{bmatrix}$$

$$I_{2_4} := D_{M_1} \cdot B_{2_3} = \begin{bmatrix} 0.00048 \\ 0.00038 \\ -0.00044 \end{bmatrix} \quad \lambda_{2_4} := I_{2_4_0} = 0.00048379 \quad B_{2_4} := \frac{1}{I_{2_4_0}} \cdot I_{2_4} = \begin{bmatrix} 1.00000 \\ 0.78900 \\ -0.90629 \end{bmatrix}$$

$$I_{2_5} := D_{M_1} \cdot B_{2_4} = \begin{bmatrix} 0.00048 \\ 0.00038 \\ -0.00044 \end{bmatrix} \quad \lambda_{2_5} := I_{2_5_0} = 0.00048360 \quad B_{2_5} := \frac{1}{I_{2_5_0}} \cdot I_{2_5} = \begin{bmatrix} 1.00000 \\ 0.78890 \\ -0.90621 \end{bmatrix}$$

$$I_{2_6} := D_{M_1} \cdot B_{2_5} = \begin{bmatrix} 0.00048 \\ 0.00038 \\ -0.00044 \end{bmatrix} \quad \lambda_{2_6} := I_{2_6_0} = 0.00048357 \quad B_{2_6} := \frac{1}{I_{2_6_0}} \cdot I_{2_6} = \begin{bmatrix} 1.00000 \\ 0.78889 \\ -0.90620 \end{bmatrix}$$

$$I_{2_7} := D_{M_1} \cdot B_{2_6} = \begin{bmatrix} 0.00048 \\ 0.00038 \\ -0.00044 \end{bmatrix} \quad \lambda_{2_7} := I_{2_7_0} = 0.00048357 \quad B_{2_7} := \frac{1}{I_{2_7_0}} \cdot I_{2_7} = \begin{bmatrix} 1.00000 \\ 0.78888 \\ -0.90620 \end{bmatrix}$$

$$\frac{1}{\omega_{2_iter}^2} = \lambda_{2_7} \xrightarrow[\text{float, 8}]{\text{solve, } \omega_{2_iter}} \begin{bmatrix} -45.474701 \\ 45.474701 \end{bmatrix} \quad A_2 := B_{2_7} = \begin{bmatrix} 1.0000 \\ 0.7889 \\ -0.9062 \end{bmatrix}$$

Prethodno rešenje: $\omega_{rac_2_1} = 45.47472$ $\frac{1}{\phi_{2_0}} \cdot \phi_2 = \begin{bmatrix} 1.00000 \\ 0.78888 \\ -0.90620 \end{bmatrix}$

Matrica brisanja prvog i drugog tona

$$orto_1 := A_1^T \cdot Mc \cdot \begin{bmatrix} A_{1_3} \\ A_{2_3} \\ A_{3_3} \end{bmatrix} = 0 \xrightarrow[\text{float, 8}]{\text{float, 8}} 135.2538 \cdot A_{3_3} + 104.66315 \cdot A_{2_3} + 40.0 \cdot A_{1_3} = 0.0$$

$$orto_2 := A_2^T \cdot Mc \cdot \begin{bmatrix} A_{1_3} \\ A_{2_3} \\ A_{3_3} \end{bmatrix} = 0 \xrightarrow[\text{float, 8}]{\text{float, 8}} -27.186042 \cdot A_{3_3} + 31.555399 \cdot A_{2_3} + 40.0 \cdot A_{1_3} = 0.0$$

$$A_{1_3}(A_{2_3}, A_{3_3}) := orto_1 \xrightarrow[\text{float, 8}]{\text{solve, } A_{1_3}} -3.381345 \cdot A_{3_3} - 2.6165788 \cdot A_{2_3}$$

$$A_2^T \cdot Mc \cdot \begin{bmatrix} A_{1_3}(A_{2_3}, A_{3_3}) \\ A_{2_3} \\ A_{3_3} \end{bmatrix} = 0 \xrightarrow[\text{float, 8}]{\text{float, 8}} -162.43984 \cdot A_{3_3} - 73.107753 \cdot A_{2_3} = 0.0$$

$$A_{2_3}(A_{3_3}) := A_2^T \cdot Mc \cdot \begin{bmatrix} A_{1_3}(A_{2_3}, A_{3_3}) \\ A_{2_3} \\ A_{3_3} \end{bmatrix} = 0 \xrightarrow[\text{float, 8}]{\text{solve, } A_{2_3}} -2.2219236 \cdot A_{3_3}$$

$$A_{1_3}(A_{2_3}(A_{3_3}), A_{3_3}) \xrightarrow[\text{float, 8}]{\text{float, 8}} 2.4324932 \cdot A_{3_3}$$

$$S_2 := \begin{bmatrix} 0 & 0 & 2.43249319 \\ 0 & 0 & -2.2219236 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} A1_3 = 0 \cdot A1_3 + 0 \cdot A2_3 + 2.4325 \cdot A3_3 \\ A2_3 = 0 \cdot A1_3 + 0 \cdot A2_3 - 2.2219 \cdot A3_3 \\ A3_3 = 0 \cdot A1_3 + 0 \cdot A2_3 + 1 \cdot A3_3 \end{array}$$

Modifikovana dinamička matrica: $D_{M_2} := D_M \cdot S_2$

Iteracije za treći ton Početna pretpostavka: $B_{0_3} := \begin{bmatrix} 1 \\ -1 \\ 0.5 \end{bmatrix}$

$$I_{3_1} := D_{M_2} \cdot B_{0_3} = \begin{bmatrix} 0.00008 \\ -0.00007 \\ 0.00003 \end{bmatrix} \quad \lambda_{3_1} := I_{3_1_0} = 0.00007675 \quad B_{3_1} := \frac{1}{I_{3_1_0}} \cdot I_{3_1} = \begin{bmatrix} 1.00000 \\ -0.91343 \\ 0.41110 \end{bmatrix}$$

$$I_{3_2} := D_{M_2} \cdot B_{3_1} = \begin{bmatrix} 0.00006 \\ -0.00006 \\ 0.00003 \end{bmatrix} \quad \lambda_{3_2} := I_{3_2_0} = 0.00006311 \quad B_{3_2} := \frac{1}{I_{3_2_0}} \cdot I_{3_2} = \begin{bmatrix} 1.00000 \\ -0.91343 \\ 0.41110 \end{bmatrix}$$

$$I_{3_3} := D_{M_2} \cdot B_{3_2} = \begin{bmatrix} 0.00006 \\ -0.00006 \\ 0.00003 \end{bmatrix} \quad \lambda_{3_3} := I_{3_3_0} = 0.00006311 \quad B_{3_3} := \frac{1}{I_{3_3_0}} \cdot I_{3_3} = \begin{bmatrix} 1.00000 \\ -0.91343 \\ 0.41110 \end{bmatrix}$$

$$\frac{1}{\omega_{3_iter}^2} = \lambda_{3_3} \xrightarrow[\text{float, 8}]{\text{solve, } \omega_{3_iter}} \begin{bmatrix} -125.88309 \\ 125.88309 \end{bmatrix} \quad A_3 := B_{3_3} = \begin{bmatrix} 1.0000 \\ -0.9134 \\ 0.4111 \end{bmatrix}$$

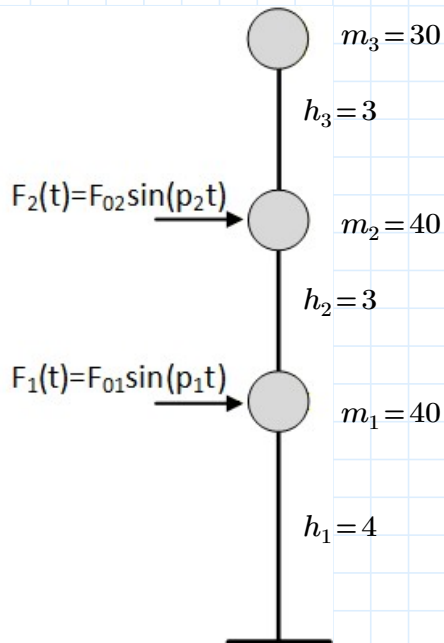
Prethodno rešenje: $\omega_{rac_2_2} = 125.88348$ $\frac{1}{\phi_{3_0}} \cdot \phi_3 = \begin{bmatrix} 1.00000 \\ -0.91343 \\ 0.41110 \end{bmatrix}$

Komentar:

Ako bi koristili, umesto dinamičke matrice, matricu $\mathbf{M}^{-1} \cdot \mathbf{K}$ prvo bi dobili svojstvene vrednosti za treći ton, pa za drugi i na kraju za prvi ton.

Primer 8. MDOF. Harmonijska sila. Numerička integracija

Za zadati dinamički model odrediti ekstremne vrednosti horizontalnih pomeranja koncentrisanih masa i ekstremne vrednosti momenata savijanja usled harmonijskih sila. Rešenje odrediti primenom modalne analize i direktne numeričke integracije. Kretanje masa samo u horizontalnom pravcu. Homogeni početni uslovi.



Podaci:

- $E = 33 \text{ GPa}$, $b/d = 30/100 \text{ cm}$
- $m_1 = 40 \text{ t}$, $m_2 = 40 \text{ t}$, $m_3 = 30 \text{ t}$
- $h_1 = 4 \text{ m}$, $h_2 = 3 \text{ m}$, $h_3 = 3 \text{ m}$
- relativno prigušenje za prva dva tona 5%
- $F_{01} = 10 \text{ kN}$, $p_1 = 7,5 \text{ rad/s}$
- $F_{02} = 10 \text{ kN}$, $p_2 = 7,5 \text{ rad/s}$

$$\zeta_1 := 0.05$$

$$\zeta_2 := 0.05$$

Matrica prigušenja

$$a_0 := 4 \cdot \pi \cdot \frac{T_{ton_1} \cdot \zeta_1 - T_{ton_2} \cdot \zeta_2}{T_{ton_1}^2 - T_{ton_2}^2} = 0.6381$$

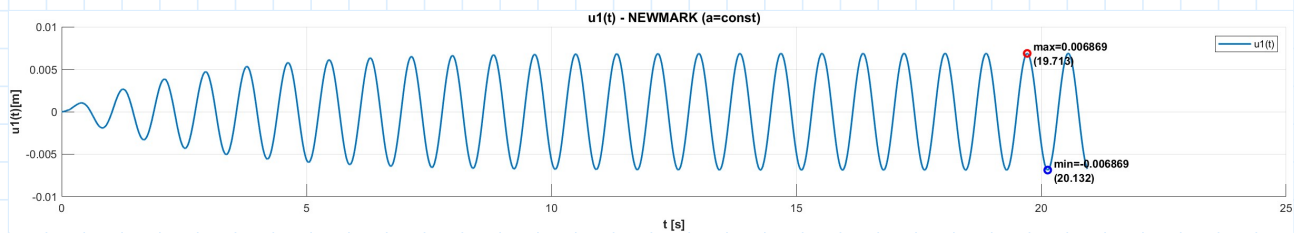
$$a_1 := \frac{1}{\pi} \cdot \frac{T_{ton_1} \cdot T_{ton_2} \cdot (T_{ton_1} \cdot \zeta_2 - T_{ton_2} \cdot \zeta_1)}{T_{ton_1}^2 - T_{ton_2}^2} = 0.0019$$

$$C := a_0 \cdot Mc + a_1 \cdot Kc = \begin{bmatrix} 706.600 & -502.549 & 141.523 \\ -502.549 & 580.060 & -207.952 \\ 141.523 & -207.952 & 111.565 \end{bmatrix}$$

Direktna numerička integracija (Matlab)

Stepen slobode 1 - ukupni odgovor - pomeranje

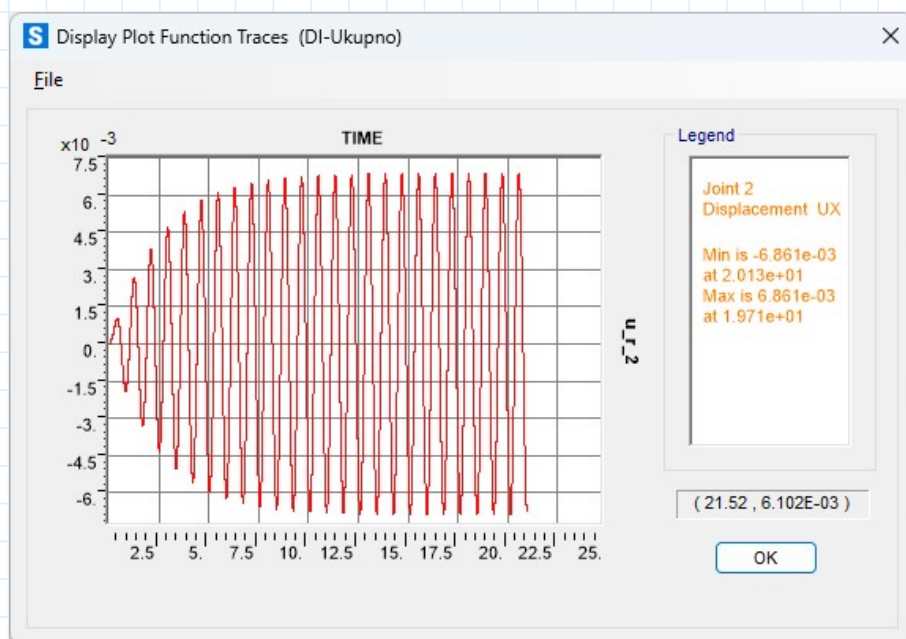
Numerička integracija - Matlab



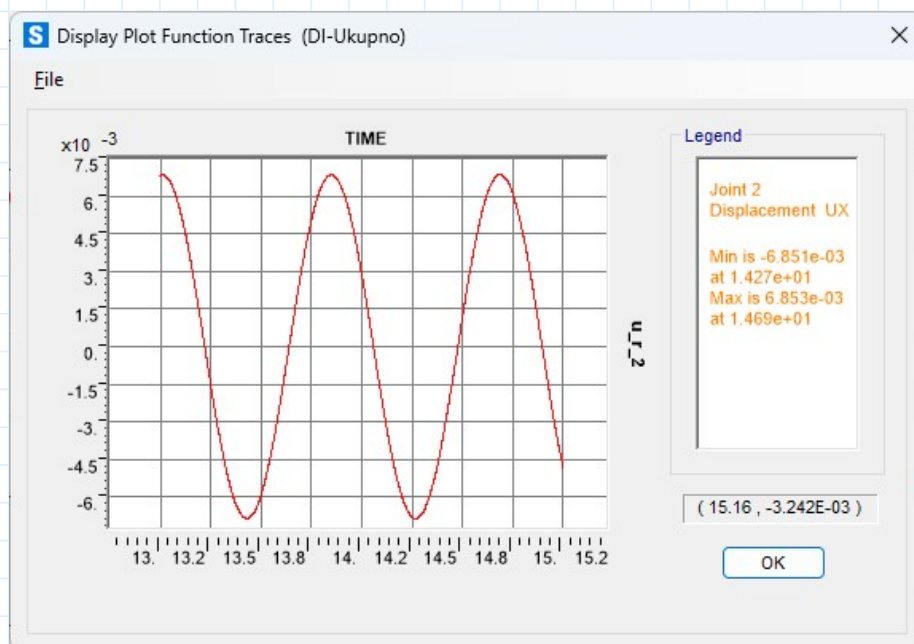
DOF = 1

u-max = 0.006869 u-min = -0.006869

SAP2000

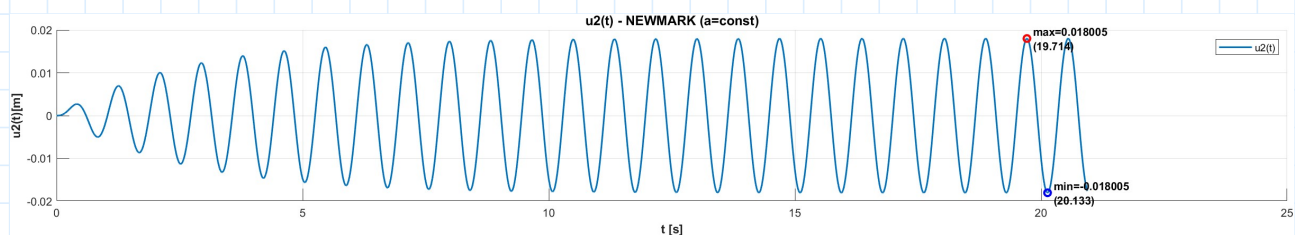


Ukupni odgovor između 13-te i 15-te s



Stepen slobode 2 - ukupni odgovor - pomeranje

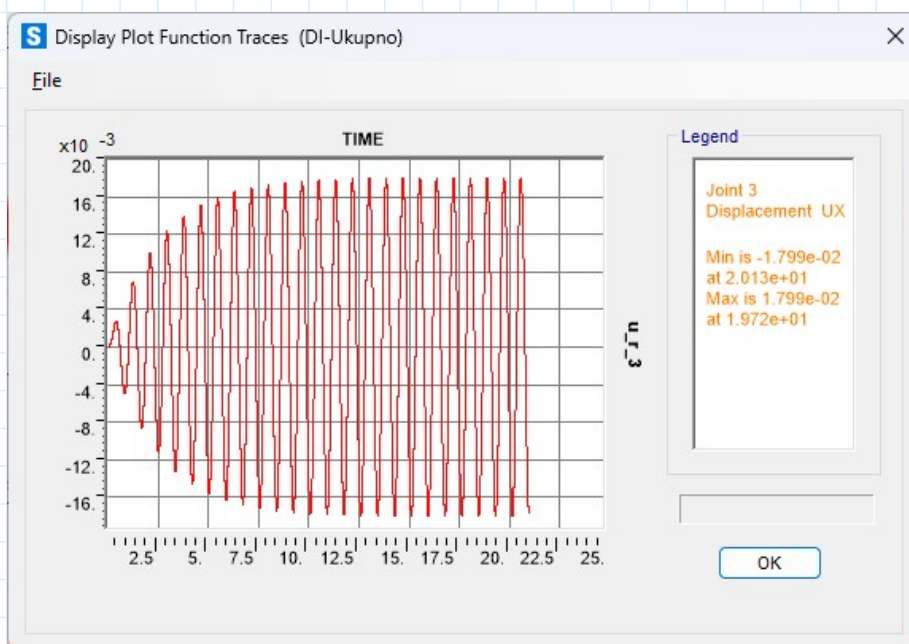
Numerička integracija - Matlab



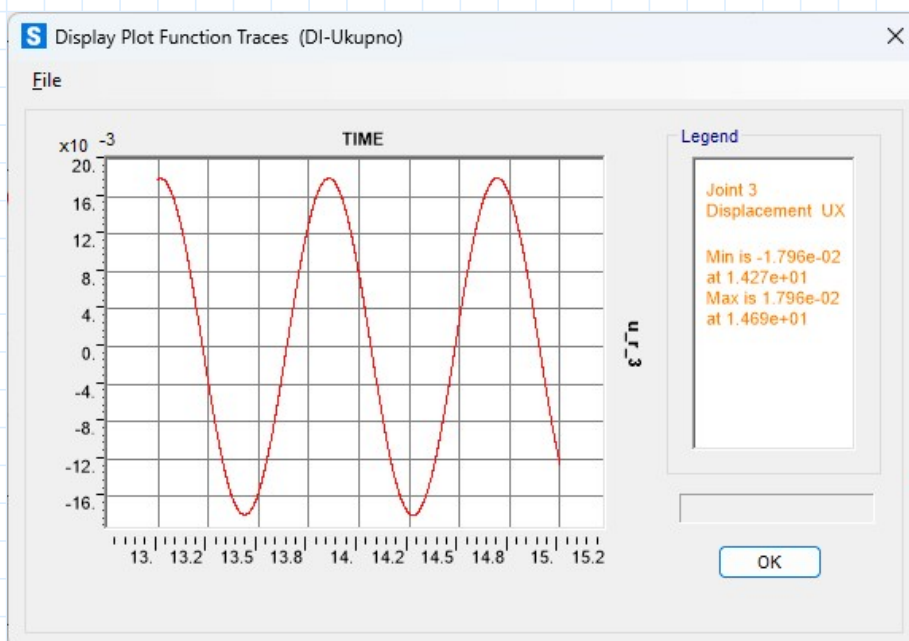
DOF = 2

u-max = 0.018005 u-min = -0.018005

SAP2000

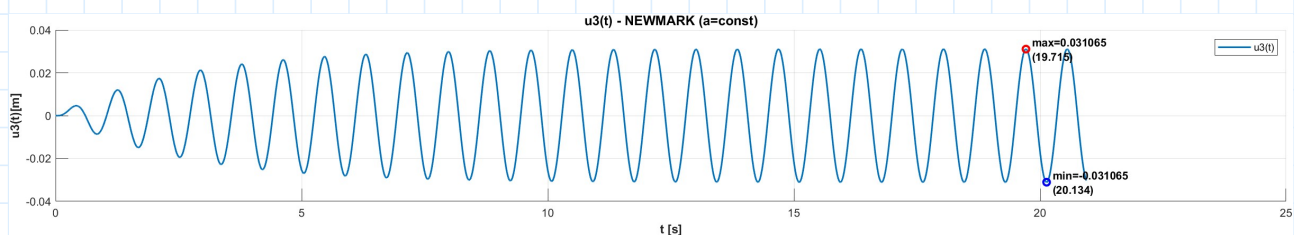


Ukupni odgovor između 13-te i 15-te s



Stepen slobode 3 - ukupni odgovor - pomeranje

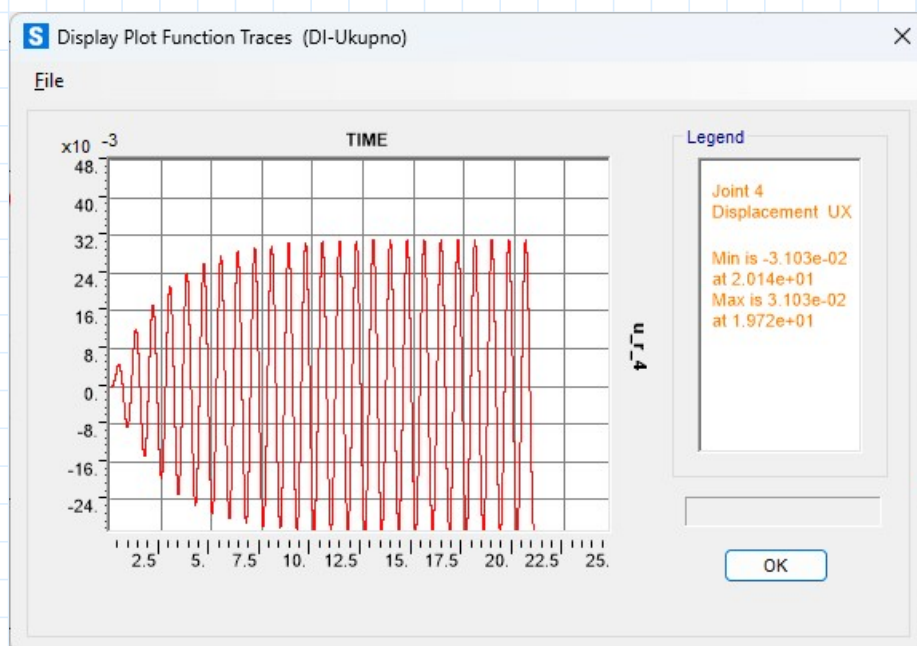
Numerička integracija - Matlab



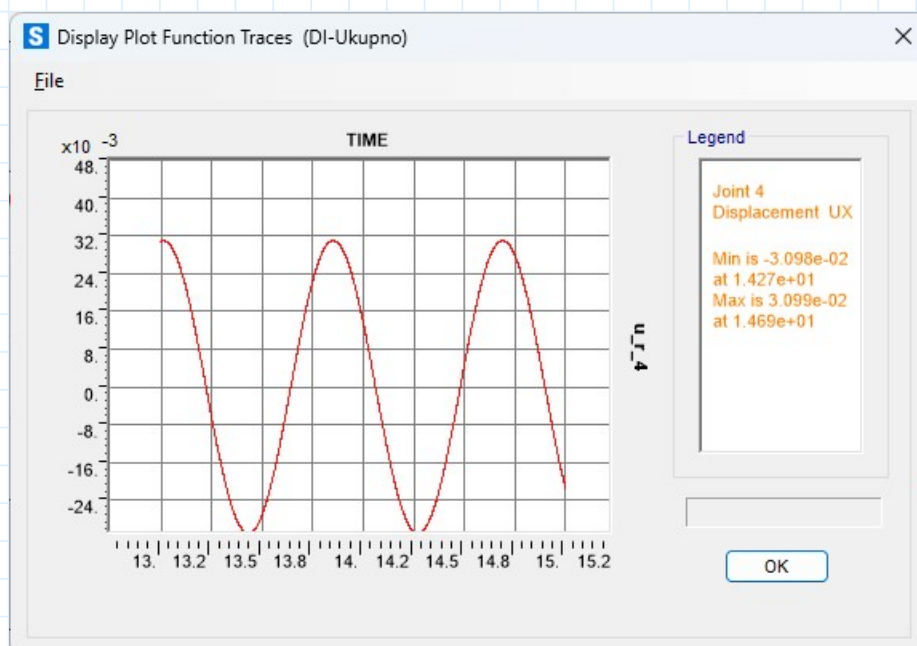
DOF = 3

$u_{\max} = 0.031065$ $u_{\min} = -0.031065$

SAP2000

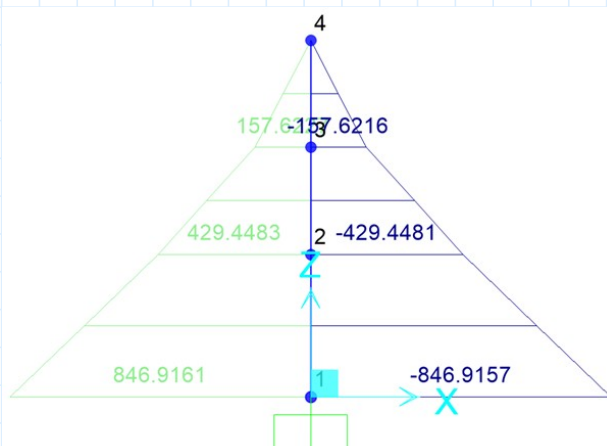


Ukupni odgovor između 13-te i 15-te s



Dijagram M - ukupni odgovor

SAP2000 - direktna numerička integracija



Numerička integracija - Matlab

Extremi dijagrama M

U nivou ukljestenja

Mu_max=847.82187430

Mu_min=-847.82146838

U nivou prve mase

M_ml_max=429.93361828

M_ml_min=-429.93347293

U nivou druge mase

M_m2_max=157.81160361

M_m2_min=-157.81048970

Modalna analiza (Matlab - numerička integracija)

Mc =

```
40.0    0.0    0.0
 0.0   40.0    0.0
 0.0    0.0   30.0
```

Kc =

```
360269.097  -265833.333   74861.111
-265833.333   293333.333  -110000.000
 74861.111  -110000.000   48888.889
```

a0 =

```
0.6381
```

a1 =

```
0.0019
```

Cc = a0*M + a1*K =

```
706.600  -502.549   141.523
-502.549   580.060  -207.952
 141.523  -207.952   111.565
```

Modalna matrica racunska iz eig

A =

```
0.22181   1.00000   1.00000
0.58037   0.78888  -0.91343
1.00000  -0.90620   0.41110
```

Modalna matrica ortonormirana

Fi =

```
0.03290   0.10569   0.11291
0.08610   0.08337  -0.10313
0.14835  -0.09577   0.04642
```

Spektralna matrica

(kvadrati svojstvenih kružnih frekvencija)

```
55.08939   0.00000   0.00000
 0.00000  2067.95045   0.00000
 0.00000   0.00000 15846.65055
```

Svojstvene kružne frekvencije

```
7.42222   0.00000   0.00000
0.00000  45.47472   0.00000
0.00000   0.00000 125.88348
```

Svojstveni periodi vibracija

```
0.84654   0.00000   0.00000
0.00000   0.13817   0.00000
0.00000   0.00000   0.04991
```

Parametri modalne jednacine:
 $w_1 = 7.422 \text{ rad/s}$; $T_1 = 0.847 \text{ s}$

Ortonormiran oblik vibracija
 0.032904
 0.086096
 0.14835

$M_{\text{gen}} = 1.000$
 $C_{\text{gen}} = 0.742$
 $K_{\text{gen}} = 55.089$
 $\text{ceta} = 0.050$

Parametri modalne jednacine:
 $w_3 = 125.883 \text{ rad/s}$; $T_3 = 0.050 \text{ s}$

Ortonormiran oblik vibracija
 0.11291
 -0.10313
 0.046416

$M_{\text{gen}} = 1.000$
 $C_{\text{gen}} = 30.596$
 $K_{\text{gen}} = 15846.651$
 $\text{ceta} = 0.122$

Parametri modalne jednacine:
 $w_2 = 45.475 \text{ rad/s}$; $T_2 = 0.138 \text{ s}$

Ortonormiran oblik vibracija
 0.10569
 0.083374
 -0.095773

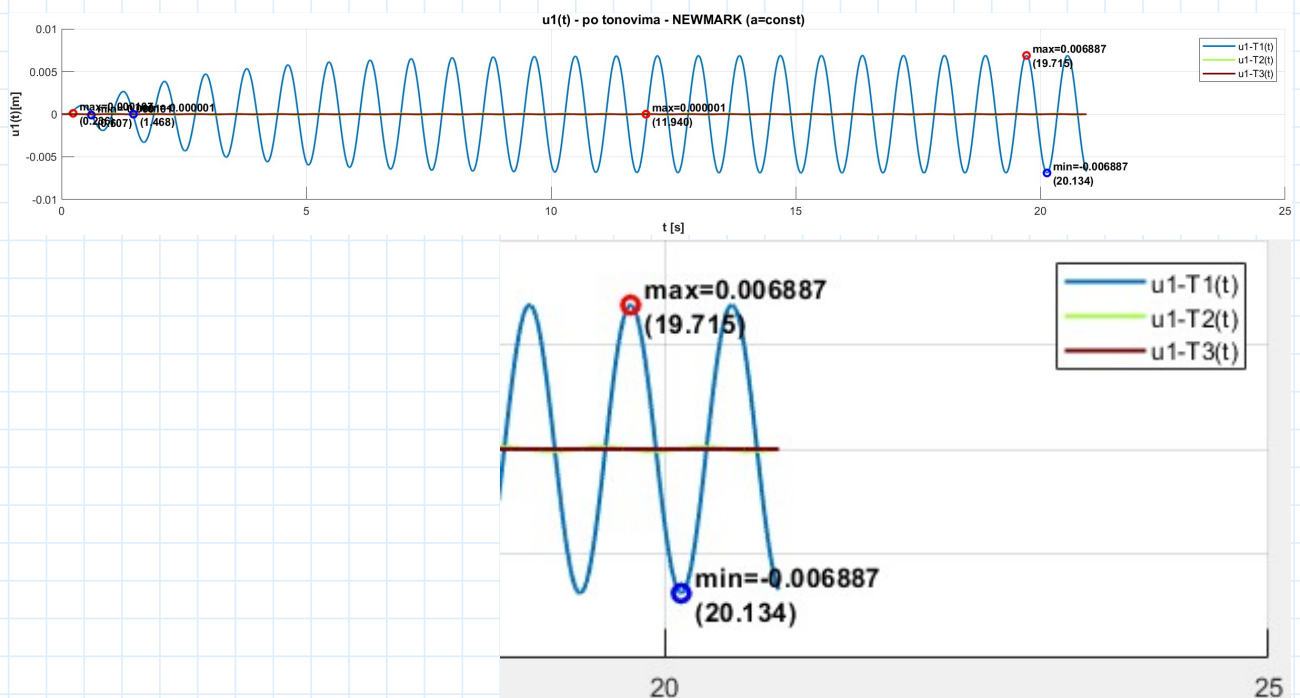
$M_{\text{gen}} = 1.000$
 $C_{\text{gen}} = 4.547$
 $K_{\text{gen}} = 2067.950$
 $\text{ceta} = 0.050$

Extremi pomeranja

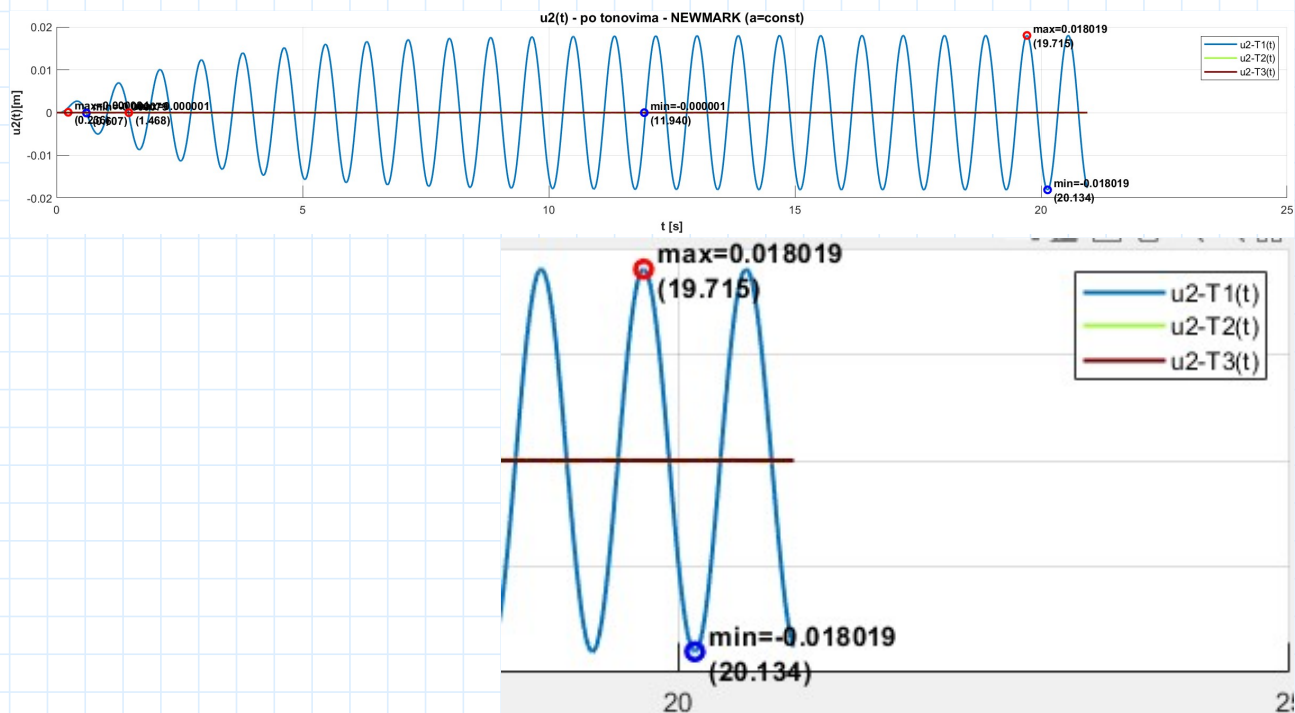
DOF = 1
 $u\text{-max} = 0.006869$ $u\text{-min} = -0.006869$
 DOF = 2
 $u\text{-max} = 0.018005$ $u\text{-min} = -0.018005$
 DOF = 3
 $u\text{-max} = 0.031065$ $u\text{-min} = -0.031065$

Udeo u ukupnom odgovoru po tonovima

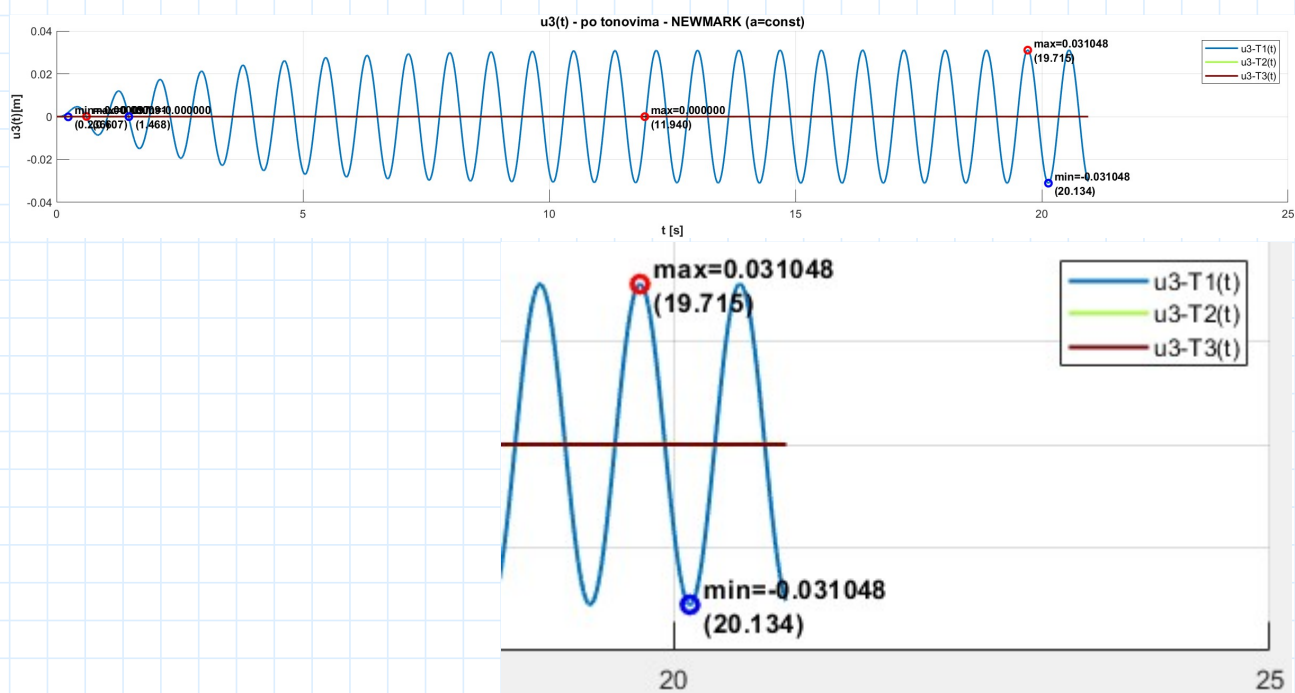
Stepen slobode kretanja 1 - u_1



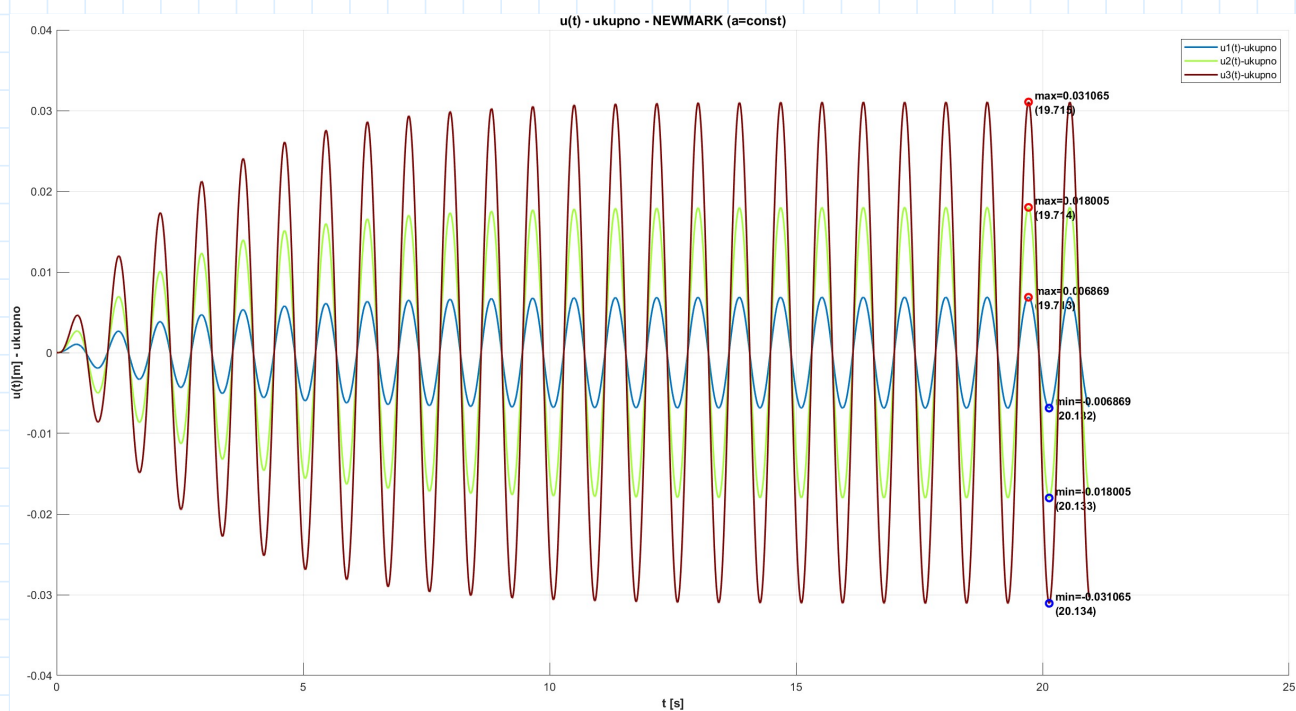
Stepen slobode kretanja 2 - u2



Stepen slobode kretanja 3 - u3

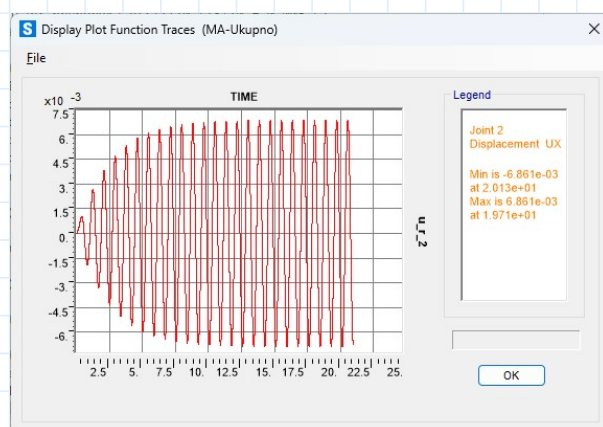


Ukupni odgovor

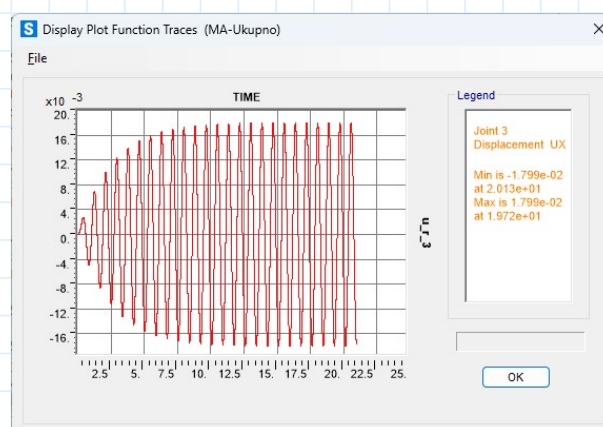


SAP2000 (ukupni odgovor)

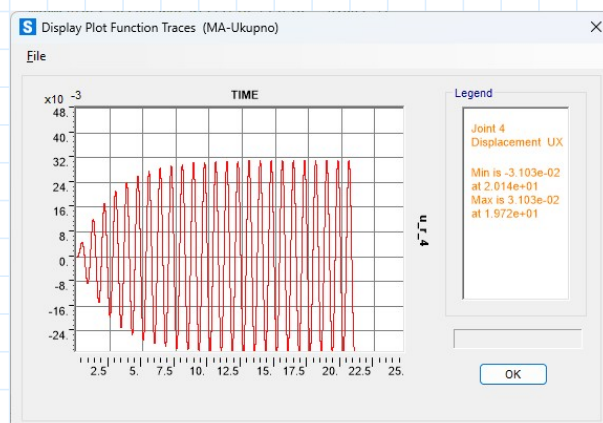
Stepen slobode 1



Stepen slobode 2

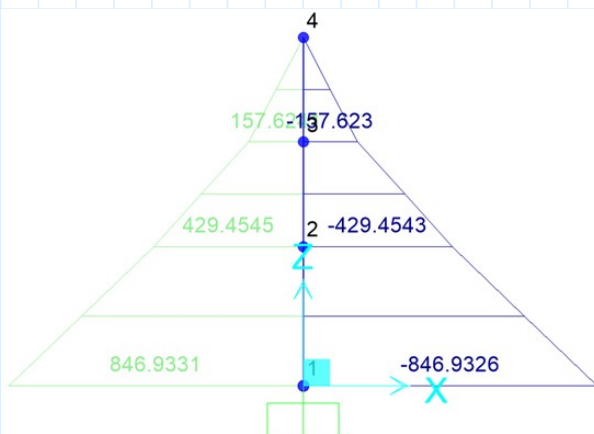


Stepen slobode 3

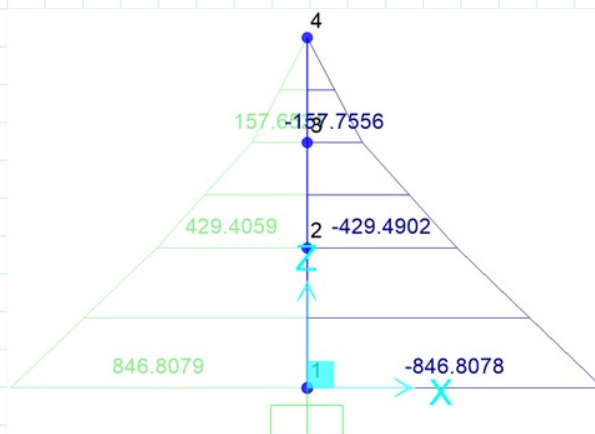


SAP2000

Dijagram M
ukupni odgovor



Dijagram M
ustaljeni odgovor



Numerička integracija - Matlab

Dijagram M
ukupni odgovor

Extremi dijagrama M

U nivou ukljestenja

Mu_max=847.82187429

Mu_min=-847.82146836

U nivou prve mase

M_m1_max=429.93361828

M_m1_min=-429.93347292

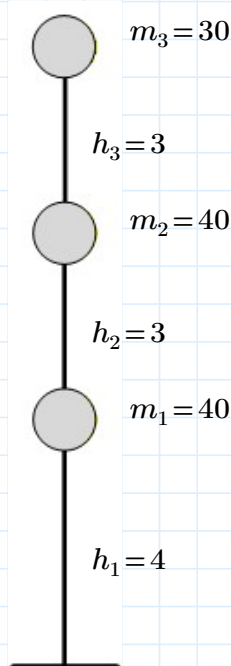
U nivou druge mase

M_m2_max=157.81160361

M_m2_min=-157.81048970

Primer 9. MDOF. Vremenska analiza odgovora. Akcelerogram

Za zadati dinamički model odrediti ekstremne vrednosti horizontalnih pomeranja koncentrisanih masa i ekstremne vrednosti momenata savijanja usled prinudnog pomeranja osnove zadanog preko akcelerograma (txt datoteka priložena uz primer). Rešenje odrediti primenom modalne analize i direktne numeričke integracije. Kretanje masa samo u horizontalnom pravcu. Homogeni početni uslovi. Karakteristike akcelerograma: interval zapisa 0.001 s i trajanje 20.0 s.



Podaci:

- $E = 33 \text{ GPa}$, $b/d = 30/100 \text{ cm}$
- $m_1 = 40 \text{ t}$, $m_2 = 40 \text{ t}$, $m_3 = 30 \text{ t}$
- $h_1 = 4 \text{ m}$, $h_2 = 3 \text{ m}$, $h_3 = 3 \text{ m}$
- relativno prigušenje za prva dva tona 5%

$$\zeta_1 := 0.05$$

$$\zeta_2 := 0.05$$

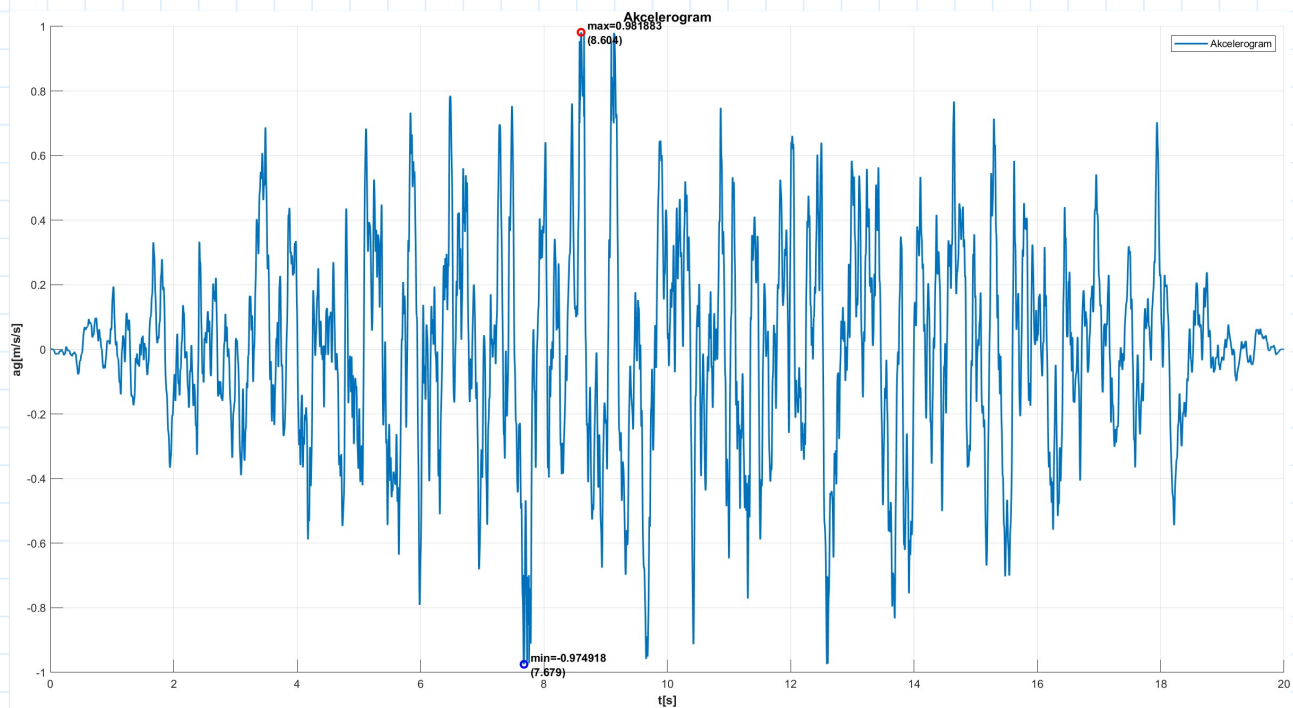
Matrica prigušenja

$$a_0 := 4 \cdot \pi \cdot \frac{T_{ton_1} \cdot \zeta_1 - T_{ton_2} \cdot \zeta_2}{T_{ton_1}^2 - T_{ton_2}^2} = 0.6381$$

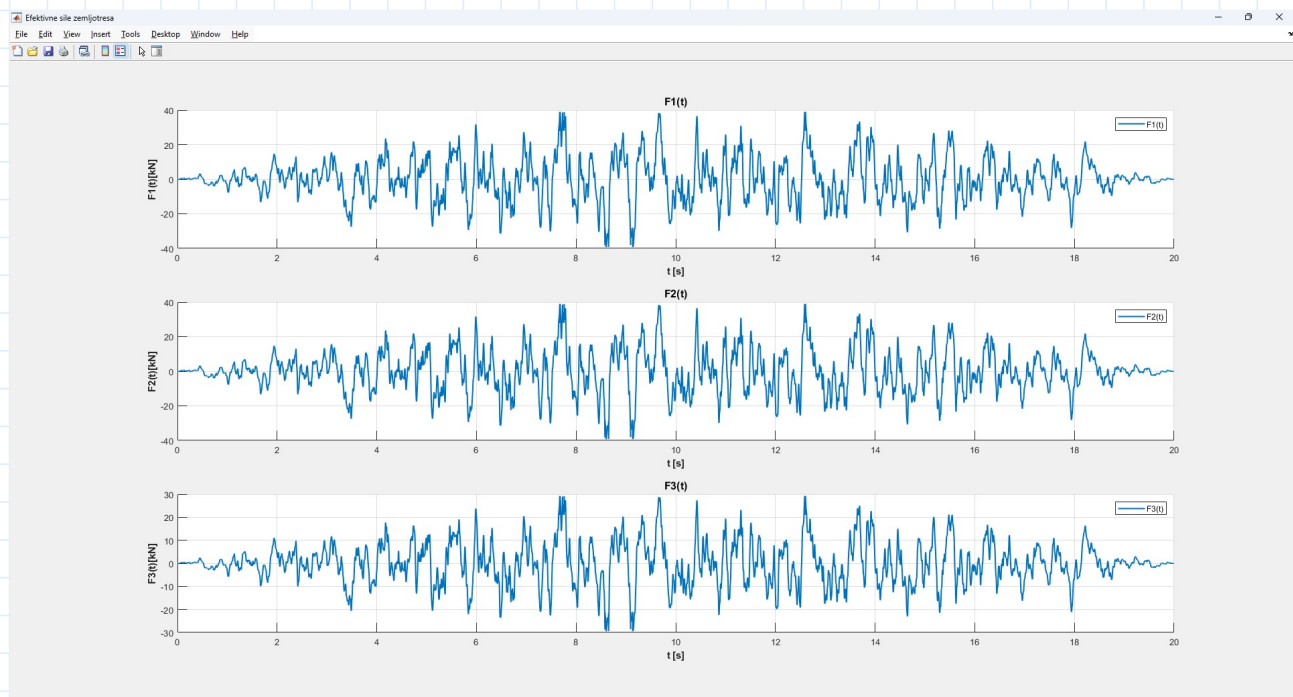
$$a_1 := \frac{1}{\pi} \cdot \frac{T_{ton_1} \cdot T_{ton_2} \cdot (T_{ton_1} \cdot \zeta_2 - T_{ton_2} \cdot \zeta_1)}{T_{ton_1}^2 - T_{ton_2}^2} = 0.0019$$

$$C := a_0 \cdot Mc + a_1 \cdot Kc = \begin{bmatrix} 706.600 & -502.549 & 141.523 \\ -502.549 & 580.060 & -207.952 \\ 141.523 & -207.952 & 111.565 \end{bmatrix}$$

Akcelerogram



Efektivne sile



Direktna numerička integracija (Matlab)

```
Mc =  
40.0    0.0    0.0  
0.0    40.0    0.0  
0.0    0.0    30.0  
Kc =  
360269.097 -265833.333 74861.111  
-265833.333 293333.333 -110000.000  
74861.111 -110000.000 48888.889
```

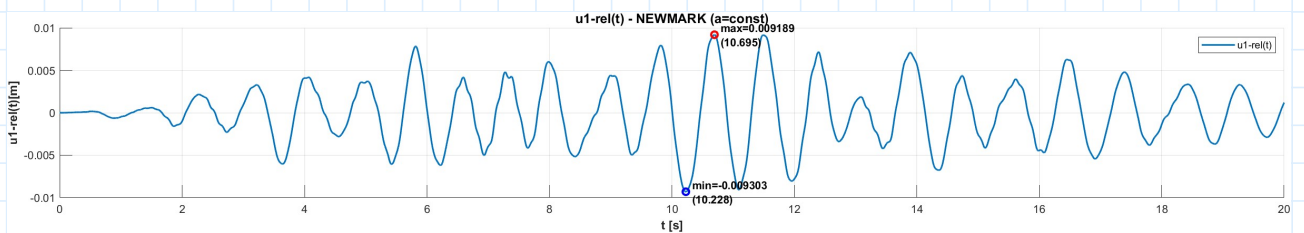
```
a0 =  
0.6381
```

```
a1 =  
0.0019
```

```
Cc = a0*M + a1*K =  
706.600 -502.549 141.523  
-502.549 580.060 -207.952  
141.523 -207.952 111.565
```

Stepen slobode 1

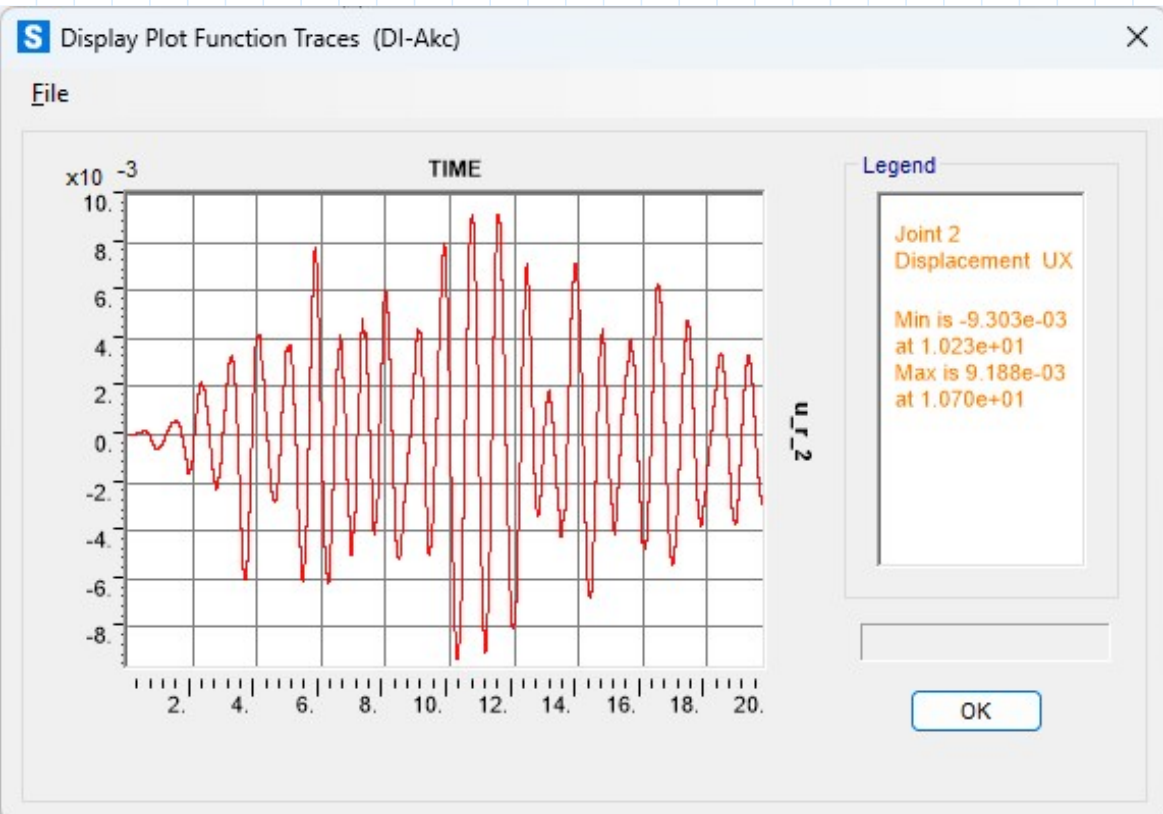
Numerička integracija - Matlab



SAP2000

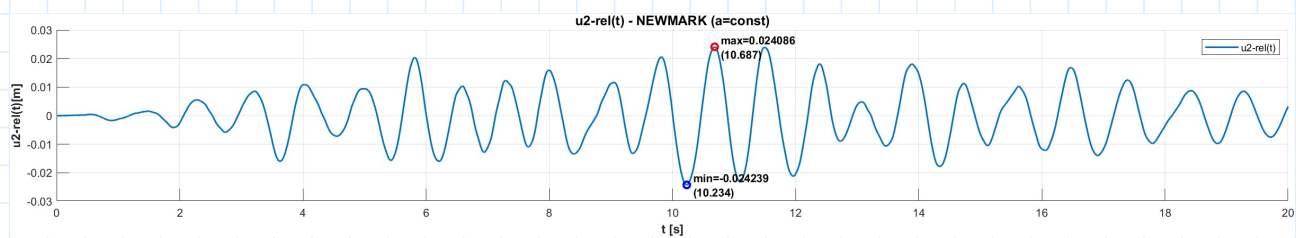
DOF = 1

u-max = 0.009189 u-min = -0.009303



Stepen slobode 2

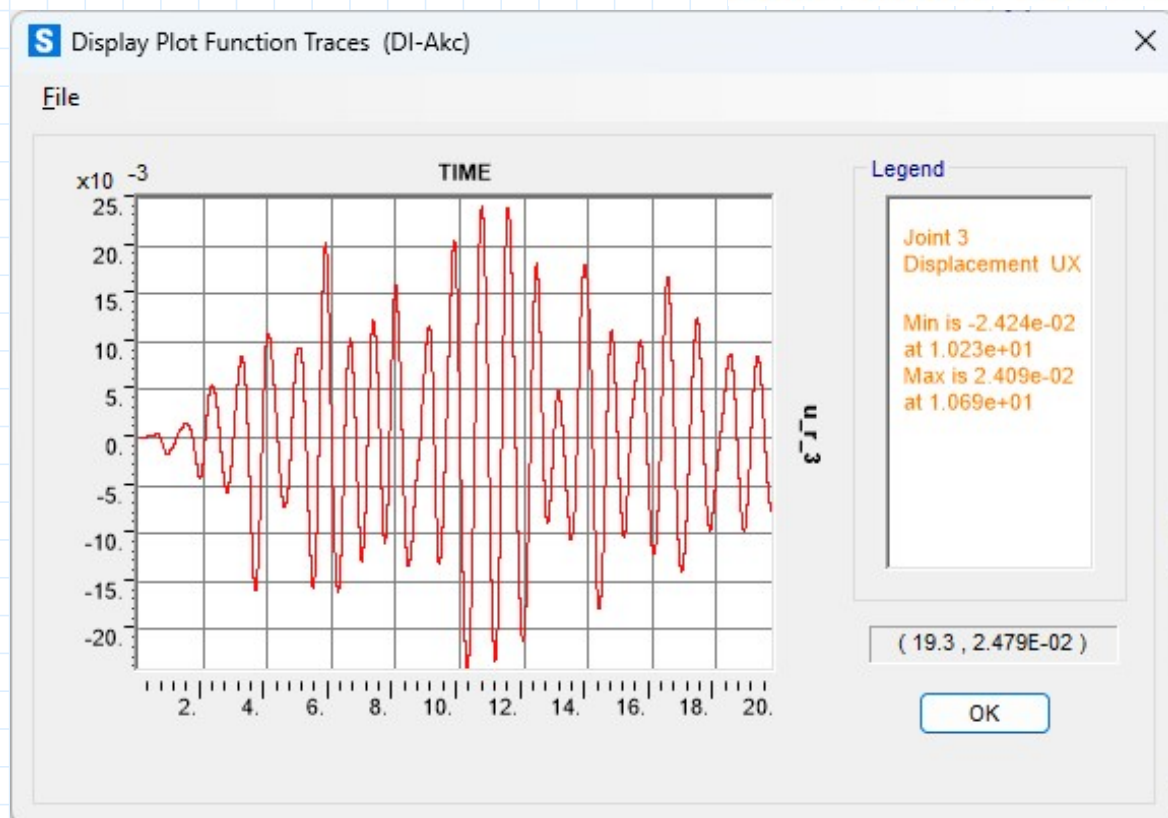
Numerička integracija - Matlab



SAP2000

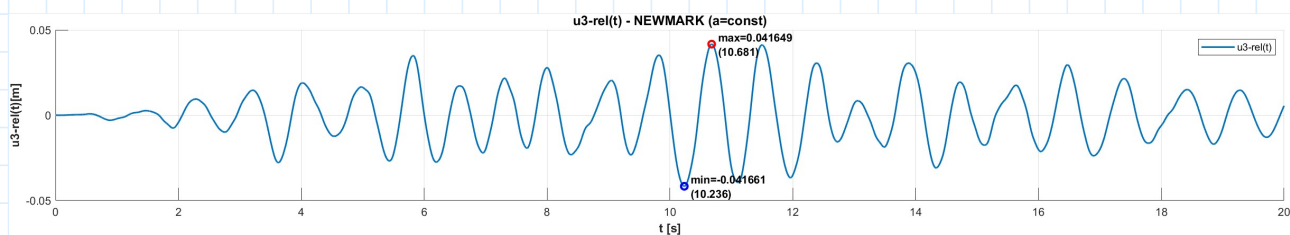
DOF = 2

u-max = 0.024086 u-min = -0.024239

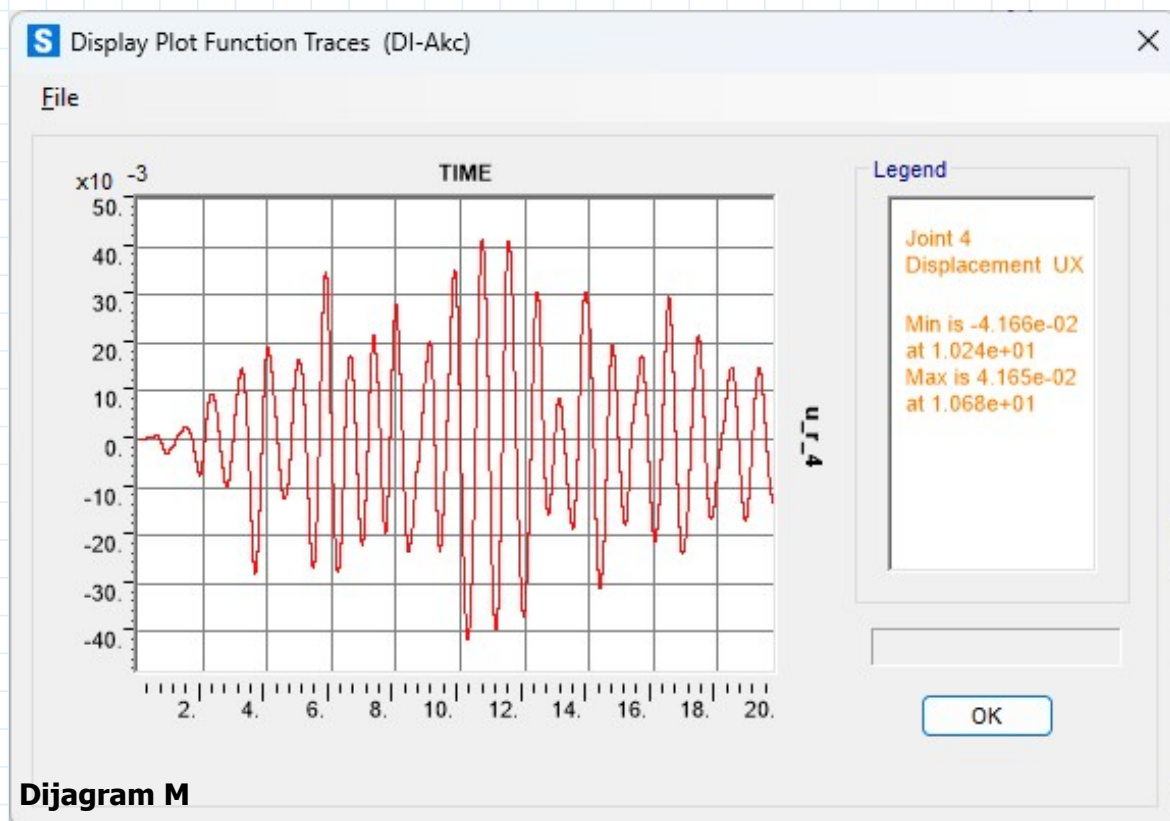


Stepen slobode 3

Numerička integracija - Matlab

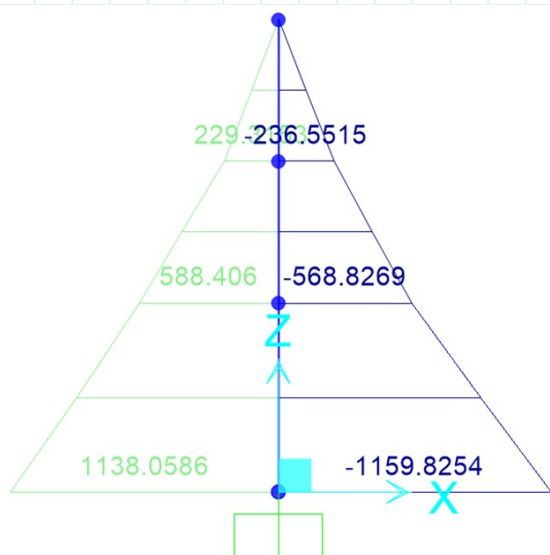


SAP2000



Dijagram M

SAP2000



Numerička integracija - Matlab

Extremi dijagrama M

U nivou ukljestenja

Mu_max=1138.07671054

Mu_min=-1159.83982298

U nivou prve mase

M_m1_max=588.41507054

M_m1_min=-568.83458210

U nivou druge mase

M_m2_max=229.31867536

M_m2_min=-236.55518561

Modalna analiza (Matlab)

```

Mc =
40.0    0.0    0.0
0.0   40.0    0.0
0.0    0.0   30.0

Kc =
360269.097  -265833.333    74861.111
-265833.333   293333.333  -110000.000
74861.111   -110000.000   48888.889

a0 =
0.6381

a1 =
0.0019

Cc = a0*M + a1*K =
706.600  -502.549   141.523
-502.549   580.060  -207.952
141.523  -207.952   111.565

Modalna matrica racunska iz eig
A =
0.22181    1.00000    1.00000
0.58037    0.78888   -0.91343
1.00000   -0.90620    0.41110

Modalna matrica ortonormirana
Fi =
0.03290    0.10569    0.11291
0.08610    0.08337   -0.10313
0.14835   -0.09577    0.04642

Spektralna matrica
(kvadrati svojstvenih kružnih frekvencija)
55.08939    0.00000    0.00000
0.00000   2067.95045    0.00000
0.00000    0.00000  15846.65055

Svojstvene kružne frekvencije
7.42222    0.00000    0.00000
0.00000   45.47472    0.00000
0.00000    0.00000   125.88348

Svojstveni periodi vibracija
0.84654    0.00000    0.00000
0.00000    0.13817    0.00000
0.00000    0.00000    0.04991

Parametri modalne jednacine:
w1 = 7.422 rad/s;   T1 = 0.847 s

Ortonormiran oblik vibracija
0.032904
0.086096
0.14835

M_gen = 1.000
C_gen = 0.742
K_gen = 55.089
ceta = 0.050

Parametri modalne jednacine:
w2 = 45.475 rad/s;   T2 = 0.138 s

Ortonormiran oblik vibracija
0.10569
0.083374
-0.095773

M_gen = 1.000
C_gen = 4.547
K_gen = 2067.950
ceta = 0.050

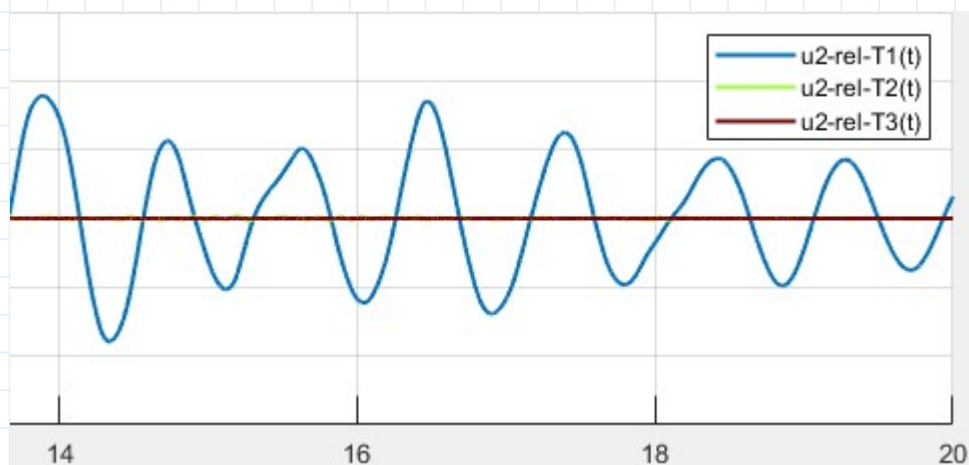
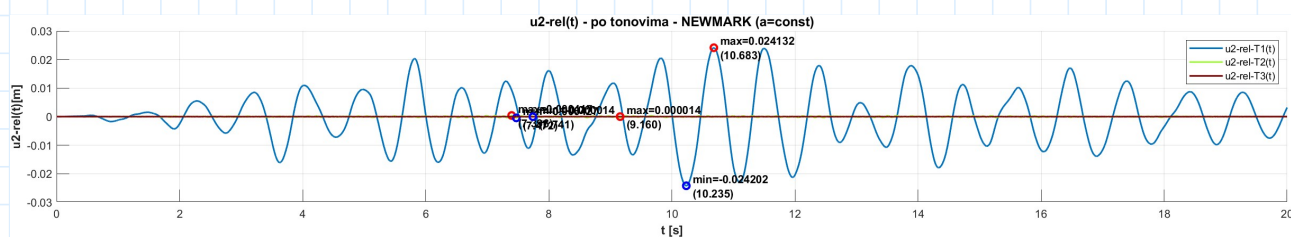
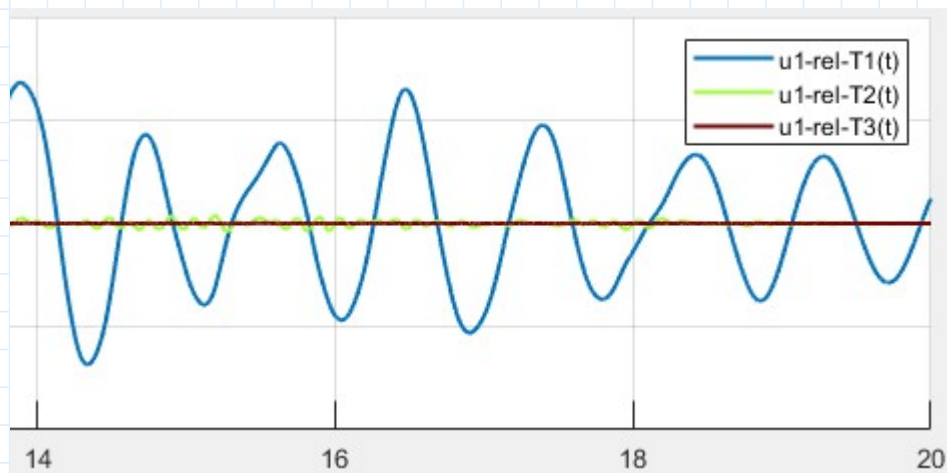
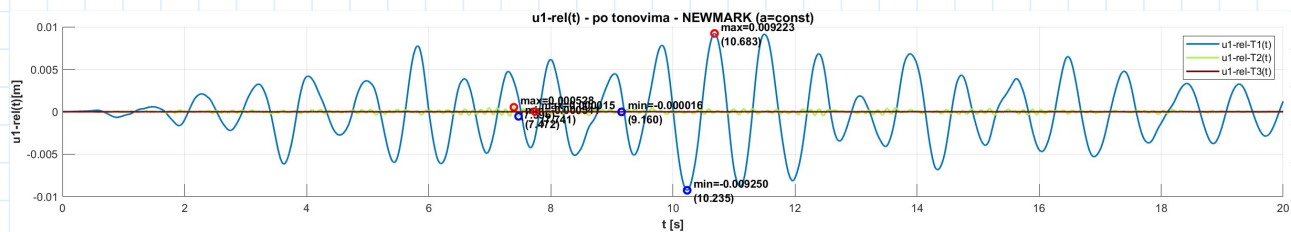
Parametri modalne jednacine:
w3 = 125.883 rad/s;   T3 = 0.050 s

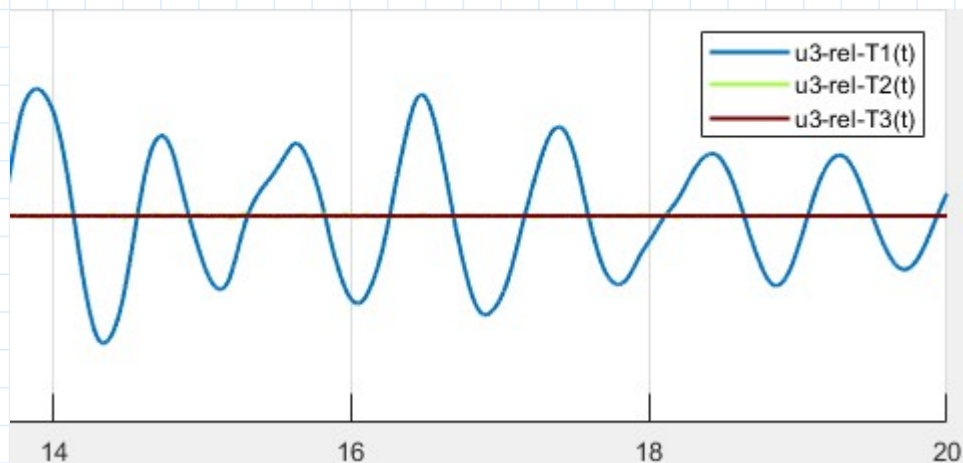
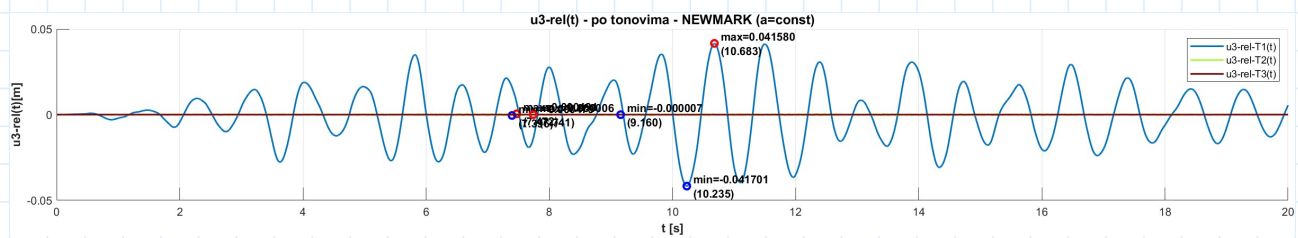
Ortonormiran oblik vibracija
0.11291
-0.10313
0.046416

M_gen = 1.000
C_gen = 30.596
K_gen = 15846.651
ceta = 0.122

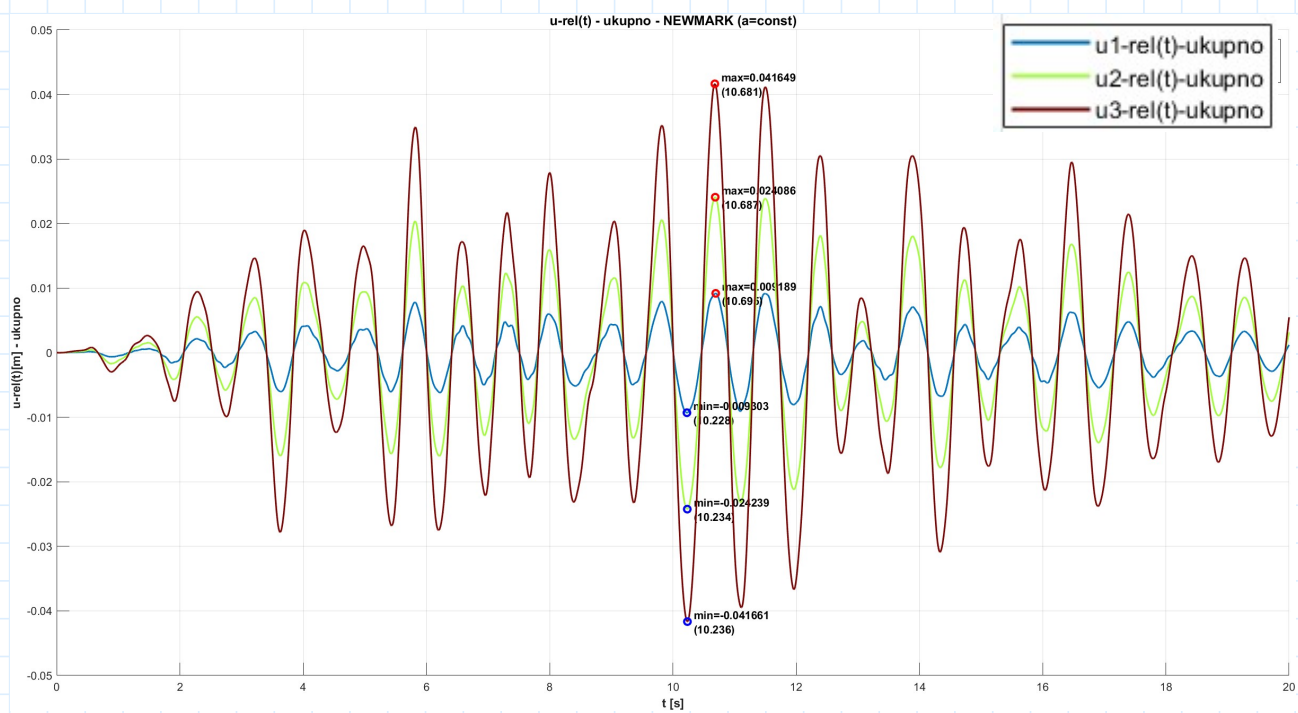
```

Udeo u ukupnom odgovoru po tonovima

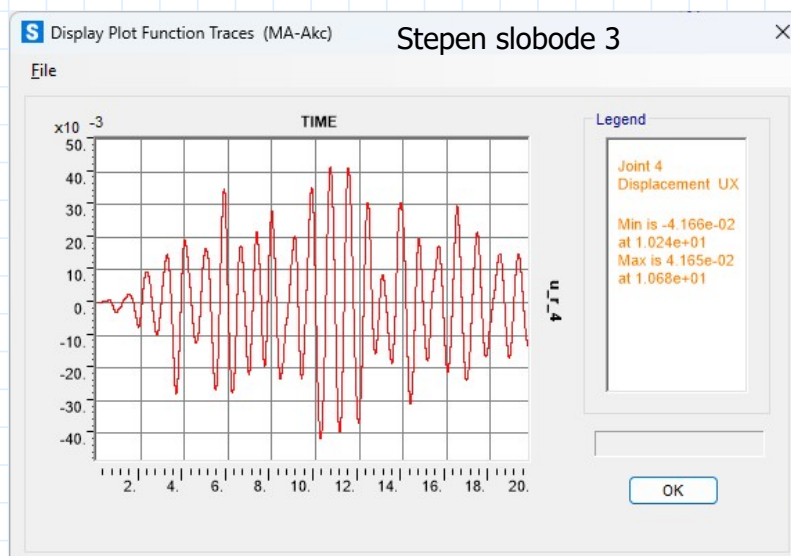
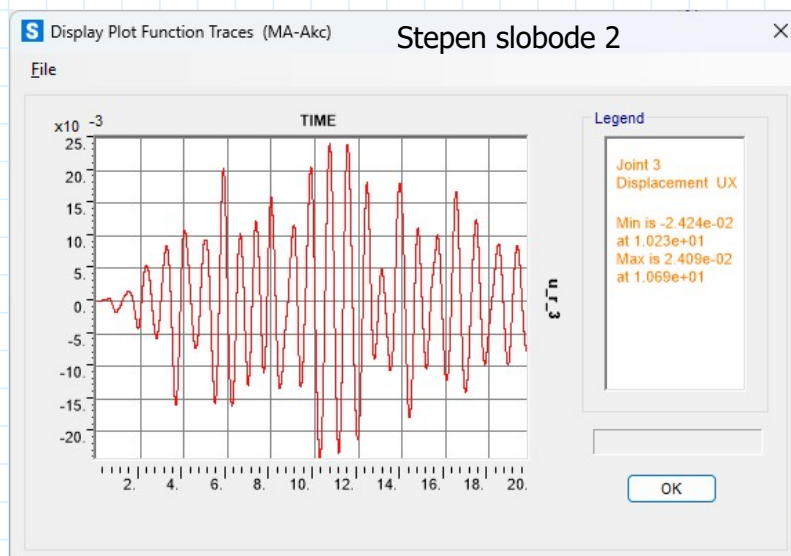
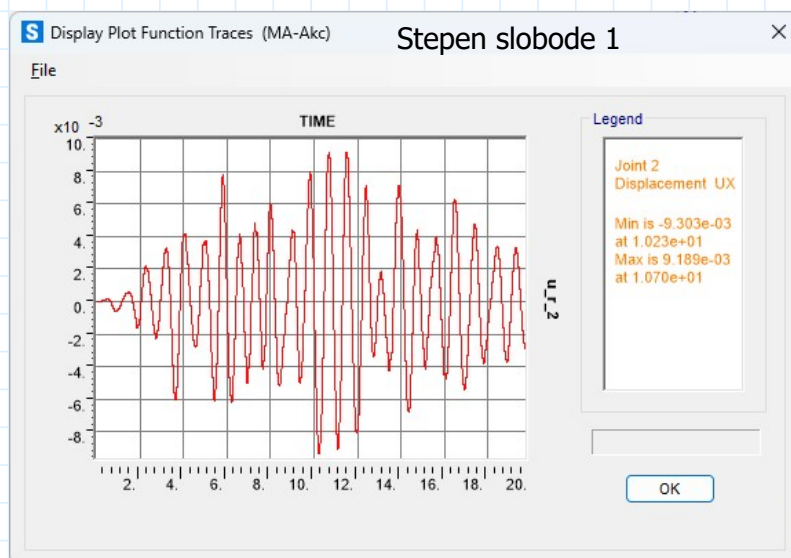




Ukupni odgovor

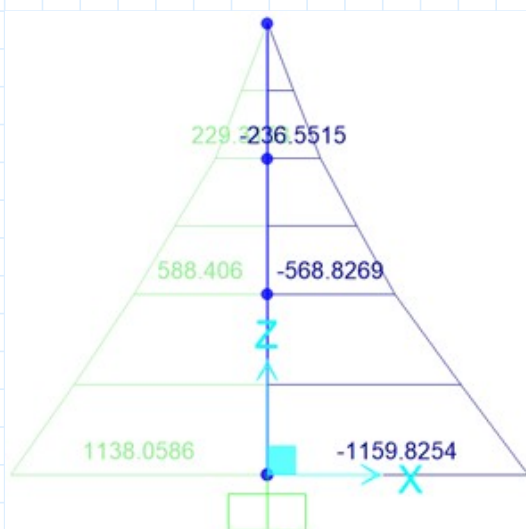


SAP2000 (ukupni odgovor)

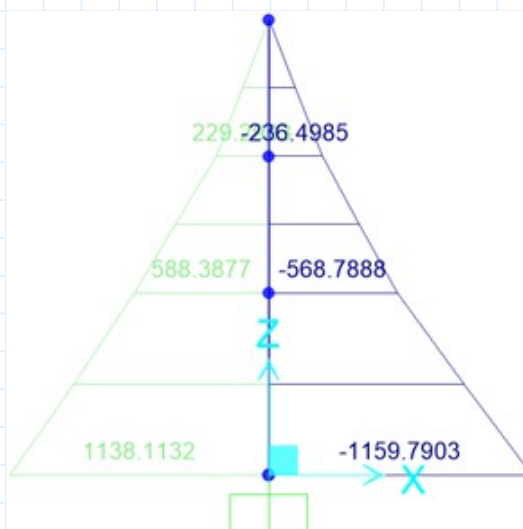


SAP2000

Dijagram M - direktna integracija



Dijagram M - modalna analiza



Modalna analiza Numerička integracija Matlab

Extremi dijagrama M

U nivou ukljescenja

Mu_max=1138.07671055

Mu_min=-1159.83982297

U nivou prve mase

M_ml_max=588.41507054

M_ml_min=-568.83458210

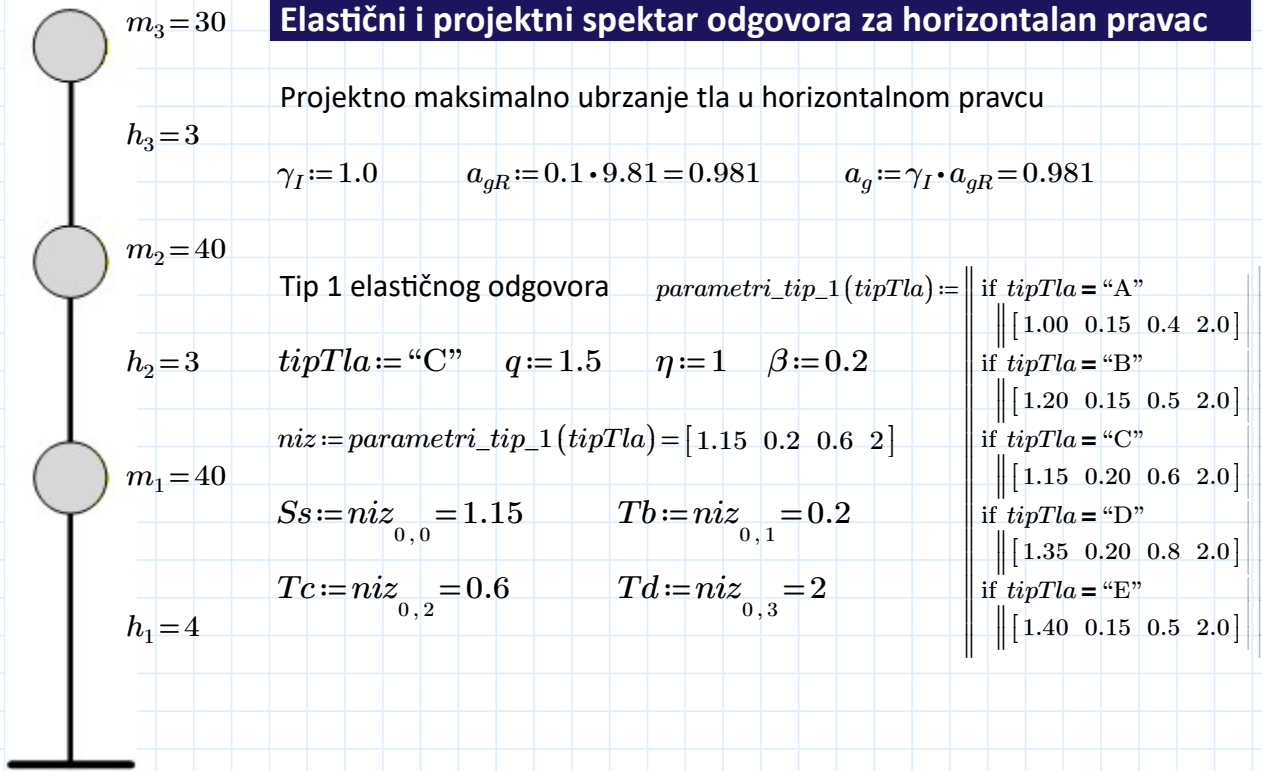
U nivou druge mase

M_m2_max=229.31867536

M_m2_min=-236.55518561

Primer 10. MDOF. Spektralna analiza odgovora

Podaci: $E = 33 \text{ GPa}$, $b/d = 30/100 \text{ cm}$, $m_1 = 40 \text{ t}$, $m_2 = 40 \text{ t}$, $m_3 = 30 \text{ t}$, referentno maksimalno ubrzanje tla u horizontalnom pravcu $a_{gR} = 0,1g$, faktor ponašanja $q = 1.5$, tip tla C, tip elastičnog odgovora 1, faktor značaja $\gamma_I = 1.0$, relativno prigušenje 5%. Pri analizi će se koristiti homogene karakteristike poprečnih preseka



$$Se(T_{ton}) := \begin{cases} Sea \leftarrow 0.0 & \text{if } 0 \leq T_{ton} \leq Tb \\ Sea \leftarrow a_g \cdot Ss \cdot \left(1 + \frac{T_{ton}}{Tb} \cdot (\eta \cdot 2.5 - 1) \right) & \text{if } Tb < T_{ton} \leq Tc \\ Sea \leftarrow a_g \cdot Ss \cdot \eta \cdot 2.5 & \text{if } Tc < T_{ton} \leq Td \\ Sea \leftarrow a_g \cdot Ss \cdot \eta \cdot 2.5 \cdot \frac{Tc}{T_{ton}} & \text{if } Td < T_{ton} \leq 4.0 \\ Sea \leftarrow a_g \cdot Ss \cdot \eta \cdot 2.5 \cdot \frac{Tc \cdot Td}{T_{ton}^2} & \text{if } 4.0 < T_{ton} \\ Sea \leftarrow a_g \cdot Ss \cdot \eta \cdot 2.5 \cdot \frac{Tc \cdot Td}{T_{ton}^2} & \text{if } 4.0 < T_{ton} \\ \text{return } Sea & \end{cases}$$

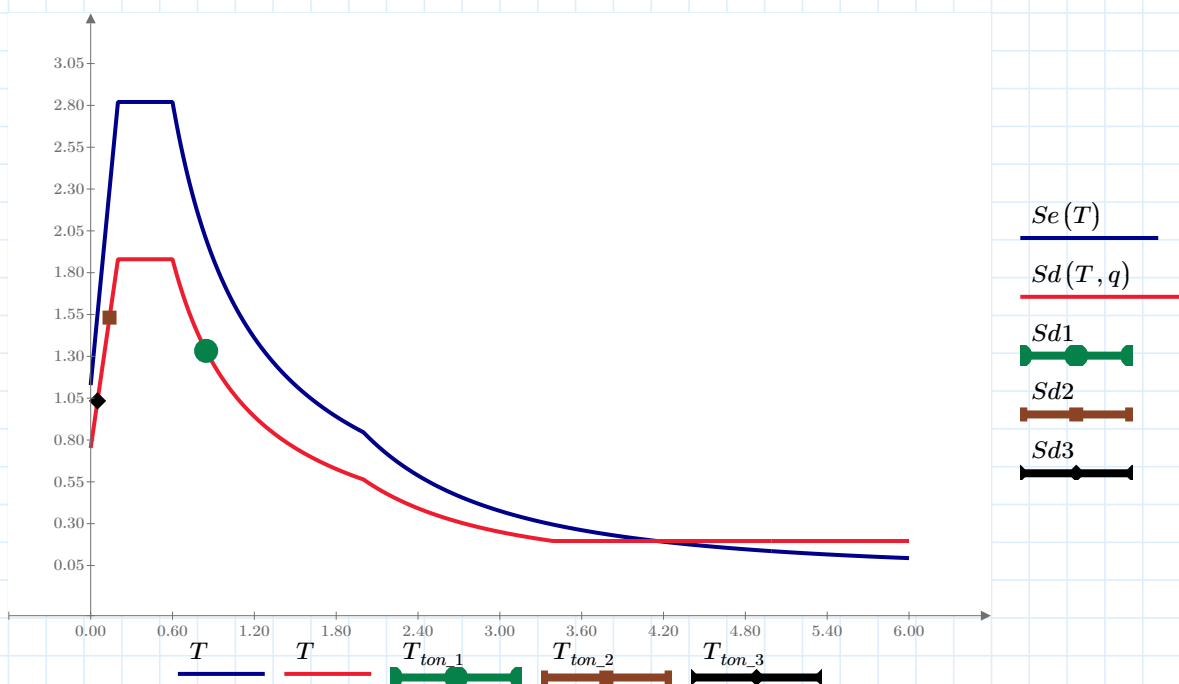
$$Sd(T_{ton}, q) := \begin{cases} Sda \leftarrow 0.0 & \text{if } 0 \leq T_{ton} \leq Tb \\ Sda \leftarrow a_g \cdot Ss \cdot \left(\frac{2}{3} + \frac{T_{ton}}{Tb} \cdot \left(\frac{2.5}{q} - \frac{2}{3} \right) \right) & \text{if } Tb < T_{ton} \leq Tc \\ Sda \leftarrow a_g \cdot Ss \cdot \frac{2.5}{q} & \text{if } Tc < T_{ton} \leq Td \\ Sda \leftarrow a_g \cdot Ss \cdot \frac{2.5}{q} \cdot \frac{Tc}{T_{ton}} & \text{if } Td < T_{ton} \leq 4.0 \\ Sda \leftarrow a_g \cdot Ss \cdot \frac{2.5}{q} \cdot \frac{Tc \cdot Td}{T_{ton}^2} & \text{if } 4.0 < T_{ton} \\ Sda \leftarrow \beta \cdot a_g & \text{if } Sda < \beta \cdot a_g \\ Sda \leftarrow \beta \cdot a_g & \text{if } Sda < \beta \cdot a_g \\ \text{return } Sda & \end{cases}$$

Projektno spektralno ubrzanje

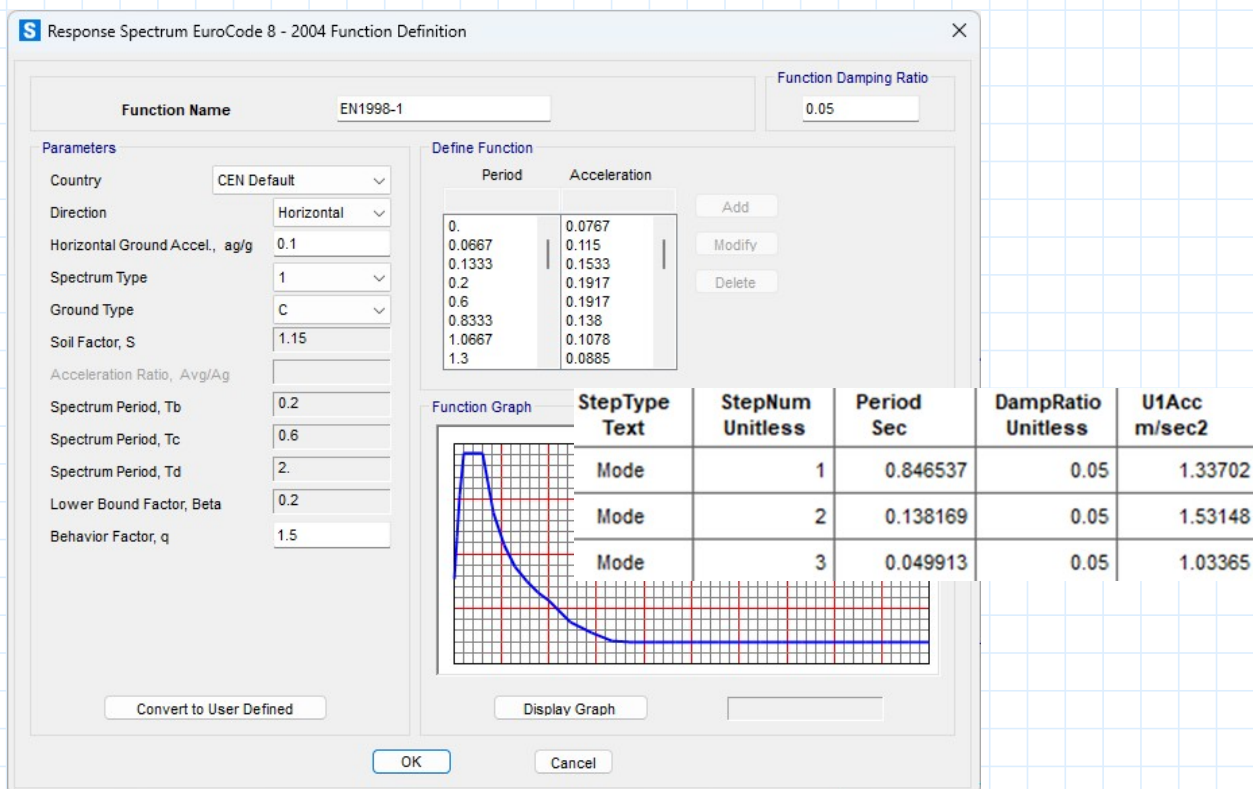
$$\beta \cdot a_g = 0.196 \quad \text{Ton 1} \quad Sd1 := Sd(T_{ton_1}, q) = 1.333 \quad \text{Ton 2} \quad Sd2 := Sd(T_{ton_2}, q) = 1.531$$

$$\text{Ton 3} \quad Sd3 := Sd(T_{ton_3}, q) = 1.034$$

$$T_{pocetak} := 0.0 \quad T_{korak} := 0.01 \quad T_{kraj} := 6.0 \quad T := T_{pocetak}, T_{korak} \dots T_{kraj}$$



SAP2000



Vektor uticajnih koeficijenata, faktor "participacije", vektor prostorne raspodele sila, doprinos n-tog tona vektoru s

Vektor uticajnih koeficijenata: $l := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\Gamma_1 := \frac{\phi_1^T \cdot Mc \cdot l}{M_1} = 9.210$$

$$\Gamma_2 := \frac{\phi_2^T \cdot Mc \cdot l}{M_2} = 4.689$$

$$\Gamma_3 := \frac{\phi_3^T \cdot Mc \cdot l}{M_3} = 1.783$$

Vektor prostorne raspodele sila: $s := Mc \cdot l = \begin{bmatrix} 40.0 \\ 40.0 \\ 30.0 \end{bmatrix}$

Doprinos n-tog tona vektoru s:

$$\Gamma_{11} := \frac{\phi_1^T \cdot s}{M_1} = 9.210$$

$$\Gamma_{22} := \frac{\phi_2^T \cdot s}{M_2} = 4.689$$

$$\Gamma_{33} := \frac{\phi_3^T \cdot s}{M_3} = 1.783$$

$$s_1 := \Gamma_{11} \cdot Mc \cdot \phi_1 = \begin{bmatrix} 12.122 \\ 31.719 \\ 40.990 \end{bmatrix}$$

$$s_2 := \Gamma_{22} \cdot Mc \cdot \phi_2 = \begin{bmatrix} 19.823 \\ 15.638 \\ -13.473 \end{bmatrix}$$

$$s_3 := \Gamma_{33} \cdot Mc \cdot \phi_3 = \begin{bmatrix} 8.054 \\ -7.357 \\ 2.483 \end{bmatrix}$$

Efektivna masa

Ukupna masa dinamičkog modela: $Mukupno := \left\| \begin{array}{l} suma \leftarrow 0 \\ \text{for } i \in 0 \dots \text{last}(Mc^{(0)}) \\ \quad \| suma \leftarrow suma + Mc_{i,i} \\ \text{return } suma \end{array} \right\| = 110.0$

Efektivna masa za T1:

$$L1 := \phi_1^T \cdot Mc \cdot l = 9.210$$

$$Mef_1 := \frac{L1^2}{M_1} = 84.83$$

$$\frac{Mef_1}{Mukupno} = 0.771$$

Efektivna masa za T2:

$$L2 := \phi_2^T \cdot Mc \cdot l = 4.689$$

$$Mef_2 := \frac{L2^2}{M_2} = 21.99$$

$$\frac{Mef_2}{Mukupno} = 0.200$$

Efektivna masa za T3:

$$L3 := \phi_3^T \cdot Mc \cdot l = 1.783$$

$$Mef_3 := \frac{L3^2}{M_3} = 3.18$$

$$\frac{Mef_3}{Mukupno} = 0.029$$

$$M_{sum_ef} := Mef_1 + Mef_2 + Mef_3 = 110.0$$

$$\frac{M_{sum_ef}}{Mukupno} = 1.000$$

SAP2000

StepType Text	StepNum Unitless	Period Sec	UX Unitless	UY Unitless	UZ Unitless	SumUX Unitless
Mode	1	0.846537	0.77119	0	0	0.77119
Mode	2	0.138169	0.1999	0	0	0.97109
Mode	3	0.049913	0.02891	0	0	1

Projektne seizmičke sile

$$\text{Ton 1: } S1 := \Gamma_1 \cdot Mc \cdot \phi_1 \cdot Sd1 = \begin{bmatrix} 16.155 \\ 42.271 \\ 54.625 \end{bmatrix}$$

$$\text{Ton 2: } S2 := \Gamma_2 \cdot Mc \cdot \phi_2 \cdot Sd2 = \begin{bmatrix} 30.359 \\ 23.950 \\ -20.633 \end{bmatrix}$$

$$s_1 \cdot Sd1 = \begin{bmatrix} 16.155 \\ 42.271 \\ 54.625 \end{bmatrix}$$

$$s_2 \cdot Sd2 = \begin{bmatrix} 30.359 \\ 23.950 \\ -20.633 \end{bmatrix}$$

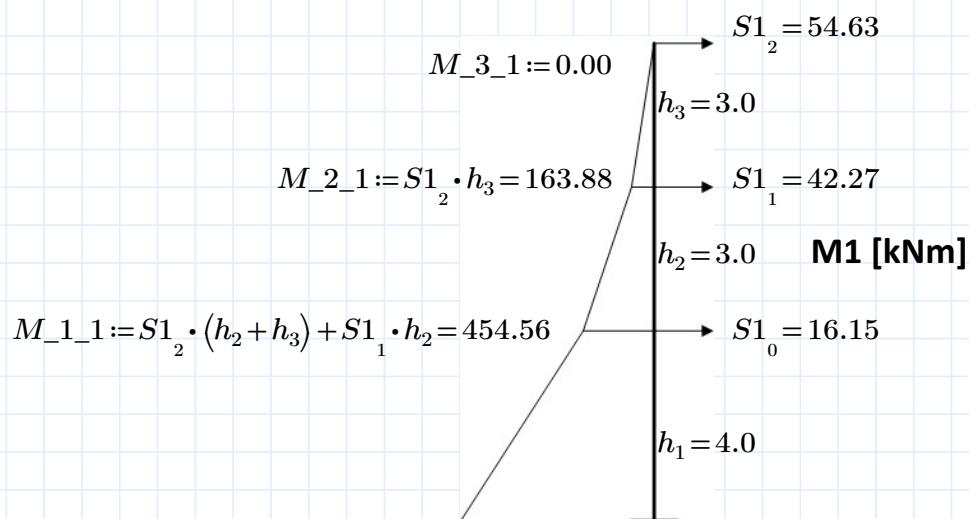
$$\text{Ton 3: } S3 := \Gamma_3 \cdot Mc \cdot \phi_3 \cdot Sd3 = \begin{bmatrix} 8.325 \\ -7.605 \\ 2.567 \end{bmatrix}$$

$$s_3 \cdot Sd3 = \begin{bmatrix} 8.325 \\ -7.605 \\ 2.567 \end{bmatrix}$$

Dijagram momenata savijanja (SRSS pravilo)

Ton 1

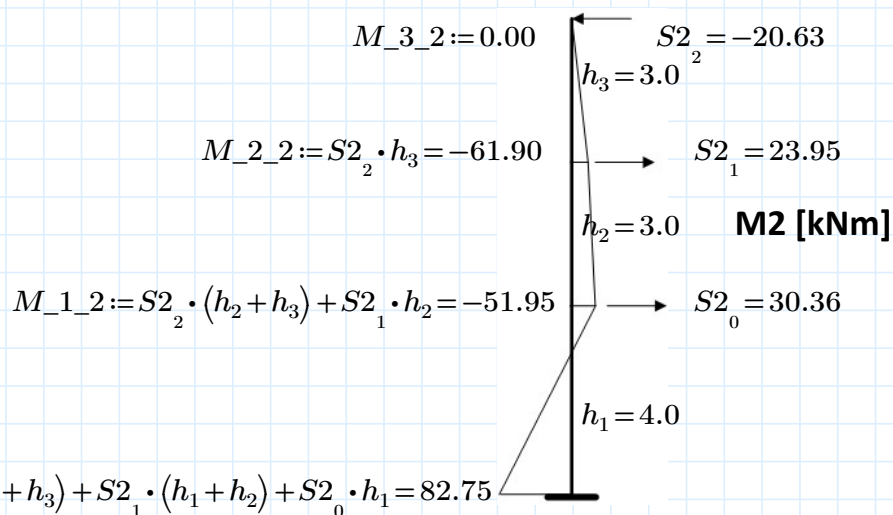
$$S1 = \begin{bmatrix} 16.15 \\ 42.27 \\ 54.63 \end{bmatrix}$$



$$M_{0_1} := S1_2 \cdot (h_1 + h_2 + h_3) + S1_1 \cdot (h_1 + h_2) + S1_0 \cdot h_1 = 906.77$$

Ton 2

$$S2 = \begin{bmatrix} 30.36 \\ 23.95 \\ -20.63 \end{bmatrix}$$



$$M_{0_2} := S2_2 \cdot (h_1 + h_2 + h_3) + S2_1 \cdot (h_1 + h_2) + S2_0 \cdot h_1 = 82.75$$

Ton 3

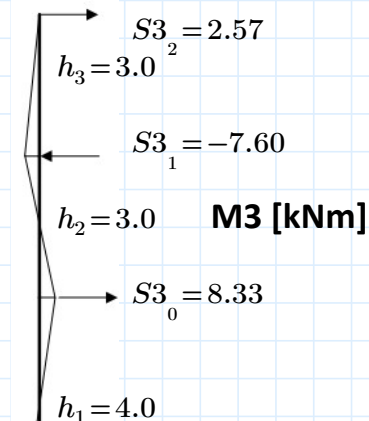
$$S3 = \begin{bmatrix} 8.33 \\ -7.60 \\ 2.57 \end{bmatrix}$$

$$M_{3_3} := 0.00$$

$$M_{2_3} := S3_2 \cdot h_3 = 7.70$$

$$M_{1_3} := S3_2 \cdot (h_2 + h_3) + S3_1 \cdot h_2 = -7.41$$

$$M_{0_3} := S3_2 \cdot (h_1 + h_2 + h_3) + S3_1 \cdot (h_1 + h_2) + S3_0 \cdot h_1 = 5.74$$



Ukupni dijagram M (SRSS pravilo)

$$\sqrt{M_{3_1}^2 + M_{3_2}^2 + M_{3_3}^2} = 0.00$$

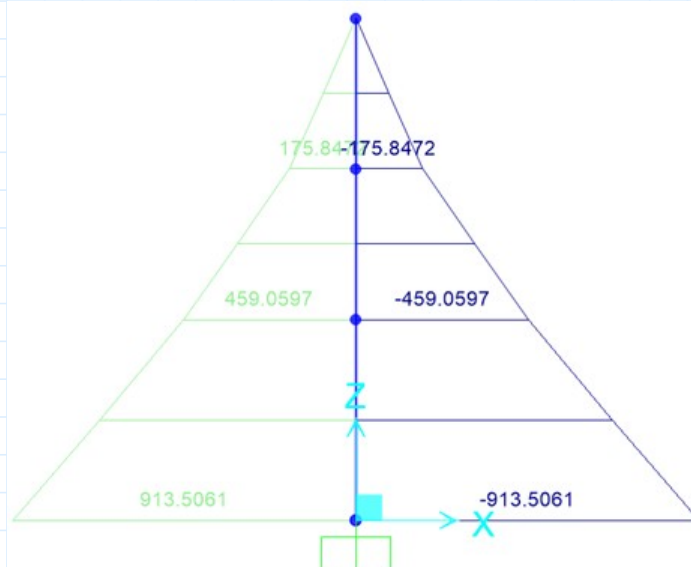
$$\sqrt{M_{2_1}^2 + M_{2_2}^2 + M_{2_3}^2} = 175.35$$

$$\sqrt{M_{1_1}^2 + M_{1_2}^2 + M_{1_3}^2} = 457.58$$

$$\sqrt{M_{0_1}^2 + M_{0_2}^2 + M_{0_3}^2} = 910.55$$

Mukupno [kNm]

SAP2000



Relativno horizontalno pomeranje u nivou masa (SRSS pravilo)

Napomena: $T_{ton_1} = 0.847 > T_c = 0.600$

$$u1 := \Gamma_1 \cdot \phi_1 \cdot \frac{Sd1}{\omega_{rac_2}^2} \cdot q = \begin{bmatrix} 0.01100 \\ 0.02877 \\ 0.04958 \end{bmatrix}$$

$$Kc^{-1} \cdot s_1 \cdot Sd1 \cdot q = \begin{bmatrix} 0.01100 \\ 0.02877 \\ 0.04958 \end{bmatrix}$$

$$u2 := \Gamma_2 \cdot \phi_2 \cdot \frac{Sd2}{\omega_{rac_1}^2} \cdot q = \begin{bmatrix} 0.00055 \\ 0.00043 \\ -0.00050 \end{bmatrix}$$

$$Kc^{-1} \cdot s_2 \cdot Sd2 \cdot q = \begin{bmatrix} 0.00055 \\ 0.00043 \\ -0.00050 \end{bmatrix}$$

$$u3 := \Gamma_3 \cdot \phi_3 \cdot \frac{Sd3}{\omega_{rac_0}^2} \cdot q = \begin{bmatrix} 0.00002 \\ -0.00002 \\ 0.00001 \end{bmatrix}$$

$$Kc^{-1} \cdot s_3 \cdot Sd3 \cdot q = \begin{bmatrix} 0.00002 \\ -0.00002 \\ 0.00001 \end{bmatrix}$$

$$u := \begin{bmatrix} \sqrt{(u1_0)^2 + (u2_0)^2 + (u3_0)^2} \\ \sqrt{(u1_1)^2 + (u2_1)^2 + (u3_1)^2} \\ \sqrt{(u1_2)^2 + (u2_2)^2 + (u3_2)^2} \end{bmatrix} = \begin{bmatrix} 0.01101 \\ 0.02878 \\ 0.04958 \end{bmatrix}$$

SAP2000

Joint Text	OutputCase	CaseType Text	StepType Text	U1 m
1	q*RS	Combination	Max	0
1	q*RS	Combination	Min	0
2	q*RS	Combination	Max	0.011047
2	q*RS	Combination	Min	-0.011047
3	q*RS	Combination	Max	0.028871
3	q*RS	Combination	Min	-0.028871
4	q*RS	Combination	Max	0.049743
4	q*RS	Combination	Min	-0.049743

